Deformed supersymmetric gauge theories from String and M-Theory

Susanne Reffert

based on work with with D. Orlando, S. Hellerman, N. Lambert
arXiv:1106.2097, 1108.0644, 1111.4811, 1204.4192, 1210.7805, 1304.3488, 1309.7350, work in progress
Introduction

In recent years, \( N=2 \) supersymmetric gauge theories and their deformations have played an important role in theoretical physics - very active research topic. Examples:

**2d gauge/Bethe correspondence (Nekrasov/Shatashvili):** relates 2d gauge theories with twisted masses to integrable spin chains.

**4d gauge/Bethe correspondence (Nekrasov/Shatashvili):** relates Omega-deformed 4d gauge theories to quantum integrable systems.

**AGT correspondence (Alday, Gaiotto, Tachikawa):** relates Omega-deformed super-Yang-Mills theory to Liouville theory.
Introduction

All these examples have two things in common:

1. A deformed supersymmetric gauge theory is linked to an integrable system.
   Relation between two very constrained and well-behaved systems that can be studied separately with different methods.
   Transfer insights from one side to the other, cross-fertilization between subjects!

2. The deformed gauge theories in question can be realized in string theory via the fluxtrap background!
   The string theory construction provides a unifying framework and a different point of view on the gauge theory problems.
Introduction

**Aim:** Realize deformed supersymmetric gauge theories via string theory. Gauge theories encode fluctuations on the world-volume of D-branes. Many parameters can be tuned by varying brane geometry.

**Here:** Deform the string theory background ("fluxtrap") into which the branes are placed (Hellerman, Orlando, S.R.)

⇒ different brane set-ups give rise to different gauge theories with seemingly unrelated deformations!

Use the fluxtrap construction to unify and meaningfully relate and reinterpret a large variety of existing results.
Introduction

Our string theoretic approach enables us moreover to generate new deformed gauge theories in a simple and algorithmic way.

Today: short overview over the many applications of the fluxtrap background and some concrete examples.

- N=2* theory in 4d
- Omega-deformed SW action from M-theory
- Alpha-, Omega-deformation and a whole SL(2,Z) worth of deformed theories

Fluxtrap background as toolbox to generate deformed gauge theories and analyze them via string theoretic methods.
Introduction
Construct the **fluxtrap background** in string theory.
Can be lifted to M-theory: **M-theory Fluxtrap**
Introduction

It captures the gauge theories with **twisted masses** of the 2d gauge/Bethe correspondence.
Introduction

We can construct the $\mathbf{N}=2^*$ theory.
Construct gravity duals of deformed $\mathbf{N}=4$ SYM
Introduction

It captures the **Omega-deformed** gauge theories of the 4d gauge/Bethe correspondence.

arXiv:1204.4192
Introduction

Can also construct Omega-deformed N=1 gauge theory.
Derive **Omega-deformed Seiberg-Witten Lagrangian**

arXiv:1304.3488
Introduction

Starting point for understanding string theory formulation of AGT correspondence.

arXiv:1210.7805
Connection to **topological string theory.**
The Fluxtrap
Background
The Fluxtrap Background

Geometrical realization of Nekrasov's construction of the equivariant gauge theory.

Start with metric with 2 periodic directions and at least a U(1)xU(1) symmetry, no B-field, constant dilaton.

Background with 4 complex independent deformation parameters:

<table>
<thead>
<tr>
<th>x</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
</tr>
</thead>
<tbody>
<tr>
<td>(ρ₁, θ₁)</td>
<td>(ρ₂, θ₂)</td>
<td>(ρ₃, θ₃)</td>
<td>(ρ₄, θ₄)</td>
<td>v</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

fluxbrane parameters

Impose identifications:

\[
\begin{align*}
\tilde{x}^8 &\sim \tilde{x}^8 + 2\pi \tilde{R}_8 \\
\tilde{x}^9 &\sim \tilde{x}^9 + 2\pi \tilde{R}_9 \\
\theta_k &\sim \theta_k + 2\pi \epsilon_k \tilde{R}_8 \\
\theta_k &\sim \theta_k + 2\pi \epsilon_k \tilde{R}_9 
\end{align*}
\]

This corresponds to the well-known Melvin or fluxbrane background.
The Fluxtrap Background

Introduce new angular variables with disentangled periodicities:
\[ \phi_k = \theta_k - \epsilon^R_k \tilde{x}^8 - \epsilon^I_k \tilde{x}^9 = \theta_k - \text{Re}(\epsilon_k \tilde{v}) \]
\[ \epsilon_k = \epsilon^R_k + i \epsilon^I_k \]
\[ \tilde{v} = \tilde{x}^8 + i \tilde{x}^9 \]

Fluxbrane metric \((T^2\)-fibration over \(\Omega\)-deformed \(\mathbb{R}^8\)):
\[
ds^2 = d\tilde{x}_0^2...7 - \frac{V^R_i V^R_j dx^i dx^j}{1 + V^R \cdot V^R} - \frac{V^I_i V^I_j dx^i dx^j}{1 + V^I \cdot V^I} \\
+ (1 + V^R \cdot V^R) \left[ dx^8 - \frac{V^R_i dx^i}{1 + V^R \cdot V^R} \right]^2 \\
+ (1 + V^I \cdot V^I) \left[ dx^9 - \frac{V^I_i dx^i}{1 + V^I \cdot V^I} \right]^2 + 2V^R \cdot V^I dx^8 dx^9
\]

Generator of rotations:
\[ V = V^R + i V^I = \epsilon_1 (x^1 \partial_0 - x^0 \partial_1) + \epsilon_2 (x^3 \partial_2 - x^2 \partial_3) + \epsilon_3 (x^5 \partial_4 - x^4 \partial_5) + \epsilon_4 (x^7 \partial_6 - x^6 \partial_7) \]

The general case breaks all supersymmetries.

Impose condition
\[
\sum_{k=1}^N \pm \epsilon_k = 0
\]
The Fluxtrap Background

T-dualize along torus directions and take decompactification limit to discard torus momenta:

**Fluxtrap background**

Before T-duality, locally, the metric was still flat, but some of the rotation symmetries were broken globally.

Bulk fields after T-duality (case $V^R \cdot V^I = 0$, $\epsilon_1 \in \mathbb{R}$, $\epsilon_2 \in i \mathbb{R}$, $\epsilon_3 = \epsilon_4 = 0$):

not anymore flat

$$ds^2 = d\rho_1^2 + \frac{\rho_1^2 d\phi_1^2 + dx_8^2}{1 + \epsilon_1^2 \rho_1^2} + d\rho_2^2 + \frac{\rho_2^2 d\phi_2^2 + dx_9^2}{1 + \epsilon_2^2 \rho_2^2} + \sum_{k=4}^7 (dx^k)^2,$$

$$B = \epsilon_1 \frac{\rho_1^2}{1 + \epsilon_1^2 \rho_1^2} d\phi_1 \wedge dx_8 + \epsilon_2 \frac{\rho_2^2}{1 + \epsilon_2^2 \rho_2^2} d\phi_2 \wedge dx_9,$$

$$e^{-\Phi} = \frac{\sqrt{\alpha'} e^{-\Phi_0}}{R} \sqrt{(1 + \epsilon_1^2 \rho_1^2)(1 + \epsilon_2^2 \rho_2^2)}$$

B-field has appeared creates a potential that localizes instantons
The Fluxtrap Background

Study resulting geometry.

Space splits into

\[ M_{10} = M_3(\epsilon_1) \times M_3(\epsilon_2) \times \mathbb{R}^4 \]

\[ \mathbb{R} \langle x_8 \rangle \rightarrow M_3(\epsilon_1) \]

R-foliation over the cigar

cigar \( \langle \rho_1, \phi_1 \rangle \)

\[ \mathbb{R} \times S^1 \]

The generator of rotations is bounded (by asymptotic radius).
The Fluxtrap Background

Now we want to lift to M-theory:

\[
\begin{align*}
\text{Consider only linear order in } \epsilon: & \\
\text{Consider only linear order in } \epsilon: \\
G_4 &= (dz + d\bar{z}) \wedge (ds + d\bar{s}) \wedge \omega \\
G_4 &= (dz + d\bar{z}) \wedge (ds + d\bar{s}) \wedge \omega \\
z &= x^8 + i x^9 & s &= x^6 + i x^{10} \\
\omega &= \epsilon_1 dx^0 \wedge dx^1 + \epsilon_2 dx^2 \wedge dx^3 + \epsilon_3 dx^4 \wedge dx^5
\end{align*}
\]
Deformed gauge theories

The type of deformation resulting from the fluxbrane background depends on how D-branes are placed into the fluxtrap with respect to the monodromies:

Deformation **not** on brane world-volume: mass deformation

\[
\begin{array}{cccc}
\text{fluxtrap} & \times & \times & \times \\
\text{D–brane} & & & \varepsilon_i & \varepsilon_j & \phi_i
\end{array}
\]

Deformation **on** brane world-volume: $\varOmega$-type deformation, Lorentz invariance broken

\[
\begin{array}{cccccc}
\text{fluxtrap} & \times & \times & \times & \times \\
\text{D–brane} & & & \varepsilon_i & \varepsilon_j & \times
\end{array}
\]

These two cases can be combined.
Examples: $N=2\star$ theory
\( N=2^* \) theory

M-theory

string theory

MFT

M5

NS5

Lift

M-theory

string theory

FT

D1

D2

D3

D4

NS5

NS5

NS5

N=(2,2) w. tw. masses

N=(8,8) w. real masses

N=4 w. real masses

N=2^*

\( N=2^* \) reciprocity

\( \Omega \)-def. N=1 SYM

\( \Omega \)-def. N=2 SYM

\( \Omega \)-def. N=4 SYM

reciproc. gauge th.

S-dual of \( \Omega \)-def. SYM

\( \Omega \)-def. N=2 SYM

2d

3d

4d

5d
N=2* theory

N=2* theory is obtained from N=4 SYM (4d) by giving equal masses to two of the scalar fields.

It is obtained from a D3-brane in the fluxtrap background

<table>
<thead>
<tr>
<th>x</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
</tr>
</thead>
<tbody>
<tr>
<td>fluxtrap</td>
<td>ε₁</td>
<td>ε₂</td>
<td>ε₃</td>
<td>ε₄</td>
<td>0</td>
<td>0</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>D3–brane</td>
<td>×</td>
<td>×</td>
<td>×</td>
<td>×</td>
<td>φ₁</td>
<td>φ₂</td>
<td>φ₃</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Deformation parameters (8 conserved supercharges)

ε₁ = ε₂ = 0
ε₃ = ε₄ = ε

Expand DBI action on D3 with up to two derivatives:

\[ \mathcal{L}_Ω = \frac{1}{4g_{YM}^2} \left[ F_{ij}F^{ij} + \frac{1}{2} \sum_{k=1}^{3} (\partial^i \phi_k) (\partial_i \bar{\phi}_k) + \frac{1}{2} \epsilon^2 \phi_1 \bar{\phi}_1 + \frac{1}{2} \epsilon^2 \phi_2 \bar{\phi}_2 \right] \]

Flows to N=2 in the IR (masses become infinite).
Different from Witten’s construction (global BC).
AN SL(2,\mathbb{Z}) of solutions
Alpha and Omega

Let us revisit the Omega-deformation.

**Bulk fields (IIA):**

\[ ds^{2}_{10} = \left[ \left( \eta_{\mu \nu} - \frac{U_\mu U_\nu}{\Delta^2} \right) dx^\mu dx^\nu + (dx^4)^2 + (dx^5)^2 + (dx^6)^2 + (dx^7)^2 + (dx^8)^2 + \left( \frac{dx^9}{\Delta^2} \right)^2 \right], \]

\[ e^\phi = \Delta^{-1}, \]

\[ B = -\frac{1}{\Delta^2} dx^9 \wedge U, \]

\[ \Delta^2 = 1 + U_i U = \epsilon_1^2 (x_0^2 + x_1^2) + \epsilon_2^2 (x_2^2 + x_3^2) + \epsilon_3^2 (x_4^2 + x_5^2) \]

The effective action of a single D4-brane in (0,1,2,3,6) suspended between two parallel NS5s is

\[ S^{\Omega}_{D4} = -\frac{1}{g^2} \int d^4 x \left[ \frac{1}{4} F_{\mu \nu} F^{\mu \nu} + \frac{1}{2} \partial_\mu X^8 \partial^\mu X^8 + \frac{1}{2} (\partial_\mu X^9 + F_{\mu \lambda} U^\lambda)(\partial^\mu X^9 + F^{\mu \rho} U_\rho) + \frac{1}{2} (U^\lambda \partial_\lambda X^8)^2 \right] \]
Alpha and Omega

We have seen that the Omega-deformation has only the B-field in the bulk.

Perform 9-11 flip: **Alpha-deformation** has only an RR-background field

\[
\begin{align*}
    ds_{10}^2 &= \Delta \left[ \left( \eta_{\mu\nu} - \frac{U_{\mu}U_{\nu}}{\Delta^2} \right) dx^\mu dx^\nu + (dx^4)^2 + (dx^5)^2 + (dx^7)^2 + \frac{(dx^6)^2 + (dx^8)^2}{\Delta^2} + (dx^9)^2 \right], \\
    e^\phi &= \Delta^{1/2}, \\
    C^{RR} &= \frac{1}{\Delta^2} dx^6 \wedge dx^8 \wedge U.
\end{align*}
\]

Study D4-brane suspended between two parallel NS5s.

The effective gauge theory action is given by

\[
S^{A}_{D4} = -\frac{1}{g^2} \int d^4 x \left[ \frac{1}{4} F_{\mu\nu}F^{\mu\nu} + \frac{1}{2\Delta^2} \left( \partial_\mu X^8 + iU^\lambda \star F_{\mu\lambda} \right) \left( \partial^\mu X^8 + iU_\rho \star F^{\mu\rho} \right) \right. \\
\left. + \frac{1}{2} \partial_\mu X^9 \partial^\mu X^9 + \frac{1}{2\Delta^2} (U^\mu \partial_\mu X^8)^2 + \frac{1}{2} (U^\mu \partial_\mu X^9)^2 \right].
\]
An SL(2,Z) of solutions

Starting from M-theory lift as before with M5-branes:

\[
\begin{align*}
M5 &: 0 \ 1 \ 2 \ 3 \ 6 \ 10 \\
M5 &: 0 \ 1 \ 2 \ 3 \ 8 \ 9
\end{align*}
\]

Reduce to 4d: eff. theory on D4 extended between to parallel NS5s.

Reduce instead on new periodic direction \( y_2 \):

\[
\begin{pmatrix}
y_1 \\
y_2
\end{pmatrix} = \begin{pmatrix} d & c \\ b & a \end{pmatrix} \begin{pmatrix} x^6/R_6 \\ x^{10}/R_{10} \end{pmatrix} = \Lambda \begin{pmatrix} x^6/R_6 \\ x^{10}/R_{10} \end{pmatrix}, \quad ad - bc = 1
\]

Resulting background contains both B- and C-fields.

\[
g = \frac{\sqrt{d^2 R_{10}^2 + c^2 R_6^2 \Delta^2}}{R_2} \left[ \left( \delta_{mn} - \frac{U_m U_n}{\Delta^2} \right) dx^m dx^n + \frac{(dx^9)^2}{\Delta^2} \right] + \frac{R_{10}^2 R_6^2 (dy^1)^2}{R_2 \sqrt{d^2 R_{10}^2 + c^2 R_6^2 \Delta^2}},
\]

\[
B = d\frac{R_{10}}{R_2} \frac{U \wedge dx^9}{\Delta^2}, \quad e^{-\Phi} = \frac{R_2^{3/2} \Delta}{(d^2 R_{10}^2 + c^2 R_6^2 \Delta^2)^{3/4}},
\]

\[
C_1 = -d R_{10} \frac{bd R_{10}^2 + ac R_6^2 \Delta^2}{d^2 R_{10}^2 + c^2 R_6^2 \Delta^2} dy^1, \quad C_3 = -b R_{10} \frac{U \wedge dx^9 \wedge dy^1}{\Delta^2}
\]
An SL(2,Z) of solutions

Expand the DBI action of the D4-brane:

\[
S^\Lambda = -\frac{1}{g_\Lambda^2} \int d^4 x \left[ \frac{1}{4} F^{\mu\nu} F_{\mu\nu} + \frac{1}{2} (\delta^{\mu\nu} + U^\mu U^\nu) \partial_\mu X^8 \partial_\nu X^8 \\
+ \frac{g_\Lambda^2}{2\Delta g_\Delta^2} \left( \partial_\mu X^9 + d \frac{g_\Omega}{g_\Lambda} F_{\mu\nu} U^\nu - i c \frac{g_\Lambda}{g_\Lambda} \ast F_{\mu\nu} U^\nu \right)^2 + c^2 \frac{g_\Lambda^2}{2\Delta g_\Delta^2} (U^\mu \partial_\mu X^9)^2 \right] \\
+ \frac{i}{4} \text{Re}[\tau] \int d^4 x F^{\mu\nu} \ast F_{\mu\nu}
\]

\[
g_\Omega^2 = \frac{R_{10}}{R_6}, \quad g_\Lambda^2 = \frac{R_6}{R_{10}} = \frac{1}{g_\Omega^2}, \quad g_\Lambda^2 = d^2 g_\Omega^2 + c^2 g_\Lambda^2, \quad g_\Delta^2 = \frac{d^2 g_\Omega^2}{\Delta} + c^2 g_\Lambda^2 \Delta \\
\tau = \frac{a(i/g_\Omega^2)}{c(i/g_\Omega^2) + d}
\]

The identity element of SL(2,Z) corresponds to the Omega-deformation: \( g_\Lambda^2 = g_\Omega^2, \ g_\Delta^2 = g_\Omega^2 / \Delta \)

The S-element, \( S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \), leads to the Alpha-deformation.

\[
g_\Lambda^2 = g_\Lambda^2 = 1/g_\Omega^2, \ g_\Delta^2 = \Delta g_\Lambda^2 = \Delta / g_\Omega^2
\]
Examples: Omega-deformed SW action
Omega-deformed SW action

M-theory

string theory

gauge theory

N=(8,8) w. tw. masses
N=(2,2) w. tw. masses

N=4 w. real masses
N=2* w. real masses

Ω-def. N=2 SYM
Ω-def. N=4 SYM

FT

MFT

MFB

M5

Lift

Reduce

Reduce +T

topological string theory

reciproc. gauge th.
S-dual of Ω-def. SW

Ω-def. N=2 SYM

Ω-def. SW

Ω-def. N=1 SYM

N=(2,2) w. tw. masses

Ω-def. N=2 SYM

Ω-def. N=4 SYM

N=(8,8) w. tw. masses

2d
3d
4d
5d

N=(8,8) w. tw. masses

N=(2,2) w. tw. masses

Ω-def. N=2 SYM

Ω-def. N=4 SYM

Ω-def. N=1 SYM

Reduce

Reduce
Omega-deformed SW

Use **M-theory lift** of fluxtrap BG.

Witten: D4 between parallel NS5s lifts to single M5 wrapped on Riemann surface.

Embed M5-brane into fluxtrap BG.

- **Self-dual three-form** on the brane.
- Still wrapped on a Riemann surface at linear order.

Take **vector** and **scalar** equations of motion in 6d (not from an action!).

Integrate equations over Riemann surface.

4d equations of motion are **Euler-Lagrange** equations of an action.

This action reduces to the **Seiberg-Witten** action in the undeformed case.

Captures **all orders** of the 4D gauge theory.
Integration over the Riemann surface of the e.o.m. results in the 4d e.o.m. for the Omega-deformed SW theory:

Vector equation:

\[
(\tau - \bar{\tau}) \left[ \partial_\mu F_{\mu\nu} + \frac{1}{2} \partial_\mu (a + \bar{a}) \hat{\omega}_{\mu\nu} + \frac{1}{2} \partial_\mu (a - \bar{a})^* \hat{\omega}_{\mu\nu} \right] \\
+ \partial_\mu (\tau - \bar{\tau}) \left[ F_{\mu\nu} + \frac{1}{2} (a - \bar{a})^* \hat{\omega}_{\mu\nu} \right] - \partial_\mu (\tau + \bar{\tau}) \left[ F_{\mu\nu} + \frac{1}{2} (a - \bar{a}) \hat{\omega}_{\mu\nu} \right] = 0
\]

Scalar equations:

\[
(\tau - \bar{\tau}) \partial_\mu \partial_\mu a + \partial_\mu a \partial_\mu \tau + 2 \frac{d\tau}{da} (F_{\mu\nu} F_{\mu\nu} + F_{\mu\nu}^* F_{\mu\nu}) \\
+ 4 \frac{d\tau}{da} (a - \bar{a}) \hat{\omega}_{\mu\nu}^+ F_{\mu\nu} - 4 (\tau - \bar{\tau}) \hat{\omega}_{\mu\nu}^+ F_{\mu\nu} = 0 ,
\]

\[
(\tau - \bar{\tau}) \partial_\mu \partial_\mu \bar{a} - \partial_\mu \bar{a} \partial_\mu \bar{\tau} - 2 \frac{d\tau}{da} (F_{\mu\nu} F_{\mu\nu} - F_{\mu\nu}^* F_{\mu\nu}) \\
+ 4 \frac{d\tau}{da} (a - \bar{a}) \hat{\omega}_{\mu\nu}^- F_{\mu\nu} - 4 (\tau - \bar{\tau}) \hat{\omega}_{\mu\nu}^- F_{\mu\nu} = 0 .
\]
Omega-deformed SW

The vector and scalar e.o.m. are the Euler-Lagrange equations of the following Lagrangian:

generalized covariant derivative for the scalar $a$, non minimal coupling to the gauge field.

$$i \mathcal{L} = - (\tau_{ij} - \bar{\tau}_{ij}) \left[ \frac{1}{2} \left( \partial_\mu a^i + 2 \left( \frac{\tau}{\tau - \bar{\tau}} \right)_{ik} * F^{k}_{\mu \nu} \hat{U}_\nu \right) \left( \partial_\mu \bar{a}^j - 2 \left( \frac{\tau}{\tau - \bar{\tau}} \right)_{jl} * F^{l}_{\mu \nu} \hat{U}_\nu \right) 
+ \left( F^i_{\mu \nu} + \frac{1}{2} (a^i - \bar{a}^i) * \hat{\omega}_{\mu \nu} \right) \left( F^j_{\mu \nu} + \frac{1}{2} (a^j - \bar{a}^j) * \hat{\omega}_{\mu \nu} \right) \right] 
+ (\tau_{ij} + \bar{\tau}_{ij}) \left( F^i_{\mu \nu} + \frac{1}{2} (a^i - \bar{a}^i) * \hat{\omega}_{\mu \nu} \right) \left( * F^j_{\mu \nu} + \frac{1}{2} (a^j - \bar{a}^j) \hat{\omega}_{\mu \nu} \right)$$

shift in the gauge field strength

$$\omega = dU$$

For $\epsilon = 0$, this reproduces the Seiberg-Witten Lagrangian. Independent of compactification radius to IIA, which is related to gauge coupling in 4d $\rightarrow$ quantum result (all orders).
Summary
Summary

The fluxtrap construction allows us to **study different gauge theories** of interest via string theoretic methods. Omega deformation and (twisted) mass deformations have **same origin** in string theory.

The construction gives a **geometrical interpretation** for the Omega BG and its properties, such as localization etc.

The S-dual of the Omega-deformation (**Alpha-deformation**) has RR-background fields.

The two cases (Alpha, Omega) are two points in a whole **SL(2,Z) class** of deformed gauge theories.

**Open questions:**
- string-theoretical realization of the AGT correspondence
- Topological string theory from the fluxtrap BG
- construct gravity duals to deformed gauge theories
Thank you for your attention!