Logarithmic Corrections

- The leading corrections to the area law for black hole entropy are logarithmic
  \[ \delta S = \frac{1}{2} D_0 \log A. \]

- These corrections can be computed from the low energy theory: only massless fields contribute.

- In some situations the corrections give non-trivial support for a known microscopic description.

- In other situations they offer clues to the nature of the unknown microscopic theory.
Updates in v. 2.0

In principle: computations are straightforward applications of techniques from the 70’s.

In recent years, Sen (and collaborators) did what we do, and more.

In practice: computations are cumbersome and intransparent.

Updates in v 2.0 focus on short-cuts that add clarity:

- Interactions with background gravity and graviphoton: employ AdS/CFT, specifically organize fluctuations as chiral primaries.
- Contributions from on-shell states only (no ghosts).
- Remnant of unphysical states: simple boundary states.
- Careful with 4D zero-modes (done incorrectly until recent years).

Setting

- Consider matter in a general theory with $\mathcal{N} \geq 2$ SUSY.

- In terms of $\mathcal{N} = 2$ fields: one SUGRA multiplet, $\mathcal{N} - 2$ (massive) gravitini, $n_V$ vector multiplets, $n_H$ hyper multiplets.

- Setting: focus on extremal black holes where it is sufficient to consider the $\text{AdS}_2 \times S^2$ near horizon region.

- The final result:
  \[ \delta S = \frac{1}{12} [23 - 11(\mathcal{N} - 2) - n_V + n_H] \log A_H. \]

- Example (relevant for microscopics): no correction in $\mathcal{N} = 4$ theory with an arbitrary number of $\mathcal{N} = 4$ matter multiplets.
Prelude: Chiral Primaries

- Massless fields in AdS\(_2 \times S^2\) organize themselves in short representations of the \(SU(2|1,1)\) supergroup.

- CFT language: consider chiral multiplets where \((h, j)\) are

\[(k, k), 2(k + \frac{1}{2}, k - \frac{1}{2}), (k + 1, k - 1)\,.

Possible values of \(k = \frac{1}{2}, 1, \frac{3}{2}, \ldots\) (\(k = \frac{1}{2}\) extra short).

- In the early days of AdS/CFT three groups independently solved linearized equations of motion and computed spectra.

- They all found the same spectrum for the \(\mathcal{N} = 2\) SUGRA multiplet.

- We used an indirect argument and found a different result.

- \textit{We are right}. 
Spherical Harmonics

- Expansion on $S^2$ of single field component with helicity $\lambda$: angular momenta $j = |\lambda|, |\lambda| + 1, \ldots$.

- Example: for a gauge field all components organize themselves into two towers with $j = 1, 2, \ldots$ and two towers with $j = 0, 1, \ldots$.

- The physical components of the vector field components organize themselves into two towers with $j = 1, 2, \ldots$.

- So: the set of physical angular momenta in each $\mathcal{N} = 2$ is unambiguous.

- Example: the $\mathcal{N} = 2$ vector multiplet has one vector field and two real scalars so the physical boson towers are: two with $j = 1, 2, \ldots$ and two with $j = 0, 1, \ldots$.

- Mixing is allowed (for same $j$) but assembly of towers into chiral multiplets uniquely determine conformal weights.
The Spectrum of Chiral Primaries

• Result: the spectrum of \((h,j)\) for all chiral primaries:

  Supergravity : \[2[(k + 2, k + 2), 2(k + \frac{5}{2}, k + \frac{3}{2}), (k + 3, k + 1)]\]
  Gravitino : \[2[(k + \frac{3}{2}, k + \frac{3}{2}), 2(k + 2, k + 1), (k + \frac{5}{2}, k + \frac{1}{2})]\]
  Vector : \[2[(k + 1, k + 1), 2(k + \frac{3}{2}, k + \frac{1}{2}), (k + 2, k)]\]
  Hyper : \[2[(k + \frac{1}{2}, k + \frac{1}{2}), 2(k + 1, k), (k + \frac{3}{2}, k - \frac{1}{2})]\]

Each tower has \(k = 0, 1, \ldots\).

• Discrepancy: previous work had one more entry in the SUGRA multiplet

  \[(1, 1), 2(\frac{3}{2}, \frac{1}{2}), (2, 0)\].

• Clarification: this field exists only as a boundary mode.
Example: Constraints for Gravity

- The graviton in $D$ dimensions has $\frac{D(D+1)}{2}$ components, $D$ gauge symmetries (from diffeomorphisms), $D$ constraints (eom’s left after gauge fixing).

- So: a graviton has $\frac{D(D-3)}{2}$ physical components.

- In 2D a graviton has $-1$ degrees of freedom so a graviton and a scalar combined has no degrees of freedom.

- Details: after gauge fixing some “equations of motion” are in fact constraints (there are no time derivatives).

- Exception: the constraint is solved by one specific spatial profile (the zero-mode on AdS$_2$) so one boundary degree of freedom can be freely specified.

- These boundary modes are physical (standard in AdS/CFT).
Quantum Fluctuations: Strategy

- All contributions from quadratic fluctuations around the classical geometry take the form

\[ e^{-W} = \int D\phi \ e^{-\phi \Lambda \phi} = \frac{1}{\sqrt{\det \Lambda}}. \]

- The quantum corrections are encoded in the heat kernel

\[ D(s) = \text{Tr} \ e^{-s\Lambda} = \sum_i e^{-s\lambda_i}. \]

- The effective action becomes

\[ W = -\frac{1}{2} \int_{\epsilon^2}^{\infty} \frac{ds}{s} D(s) = -\frac{1}{2} \int_{\epsilon^2}^{\infty} \frac{ds}{s} \int d^Dx K(s). \]

- The constant \( D_0 \) (or \( K_0 \)) we need is (essentially) the 2nd Seeley-deWitt coefficient or equivalently the trace anomaly of the EM-tensor.
Simple Heat Kernels in 2D

• The heat kernel for a scalar field on $S^2$ is elementary:

$$K^s_S(s) = \frac{1}{4\pi a^2} \sum_{k=0}^\infty e^{-sk(k+1)(2k+1)} = \frac{1}{4\pi a^2 s} \left( 1 + \frac{1}{3}s + \frac{1}{15}s^2 + \ldots \right)$$

• A massless scalar field on AdS$_2$ involves a continuous spectrum:

$$K^s_A(s) = \frac{1}{2\pi a^2} \int_0^\infty e^{-(p^2+\frac{1}{4})s} p \tanh \pi p \, dp .$$

• The local terms in the AdS$_2$ heat kernel is identical to $S^2$ except for the sign of the curvature:

$$K^s_A(s) = \frac{1}{4\pi a^2 s} \left( 1 - \frac{1}{3}s + \frac{1}{15}s^2 + \ldots \right) .$$

• The heat kernel for a fermion on $S^2$ is also elementary:

$$K^f_S(s) = \frac{1}{4\pi a^2} \sum_{k=0}^\infty e^{-s(k+1)^2(2k+2)} = \frac{1}{4\pi a^2 s} \left( 1 - \frac{1}{6}s - \frac{1}{60}s^2 + \ldots \right)$$
Simple Heat Kernels on $\text{AdS}_2 \times S^2$

• For a product space heat kernels multiply so for a scalar on $\text{AdS}_2 \times S^2$:

$$K_4^s(s) = K_S^s(s)K_A^s(s) = \frac{1}{16\pi^2 a^4 s^2} \left( 1 + \frac{1}{45}s^2 + \ldots \right) .$$

• For a Dirac fermion on $\text{AdS}_2 \times S^2$:

$$K_4^f(s) = 4K_S^f(s)K_A^f(s) = \frac{1}{4\pi^2 a^4 s^2} \left( 1 - \frac{11}{180}s^2 + \ldots \right).$$

• A benchmark for results in $\mathcal{N} = 2$ theory: a “free hyper”

$$K_4^{\text{min}}(s) = 4K_4^s(s) + K_4^f(s) = \frac{1}{4\pi^2 a^4 s^2} \cdot \frac{1}{12}s^2 .$$

• The leading $1/s^2$ singularity cancels: no cosmological constant for equal number of fermion and bosons.

• The $1/s$ order also cancels: this is an accident.
The AdS$_2$ Perspective

- The canonical heat kernel on AdS$_2$ of for a massless field.
- A field with conformal weight $h$ (mass $m^2 = h(h - 1)$) and $SU(2)$ quantum number $j$ (degeneracy $2j + 1$):
  \[
  K_A(h, j; s) = K_A(h = 1, j = 0; s) \, e^{-h(h-1)s} (2j + 1) .
  \]
- A free $4D$ boson is a tower of $2D$ bosons with $(h, j) = (k + 1, k)$ with $k = 0, 1, \ldots$ so
  \[
  K^s_4(s) = K^s_A(s) \cdot \frac{1}{4\pi a^2} \sum_{k=0}^{\infty} e^{-sk(k+1)} (2k + 1) = \frac{1}{16\pi^2 a^4 s^2} \left( 1 + \frac{1}{45} s^2 + \ldots \right) .
  \]
- The sum over the tower of AdS$_2$ fields computes the factor from the heat kernel on $S^2$. 

12
The Vector-Multiplet: Bulk

- The conformal weights for fields in supergravity are “shifted” from the free values.
- The fermions in the vector multiplet are canonical but bosons interact: this is the *attractor mechanism*.
- The “shifted” sum on $S^2$ for all four physical bosons:

\[
K_{V,b}^A(s) = \frac{2K_A^s(s)}{4\pi a^2} \sum_{k=0}^{\infty} \left( e^{-sk(k+1)}(2k + 3) + e^{-s(k+1)(k+2)}(2k + 1) \right)
\]

\[
= \frac{1}{4\pi^2 a^4 s^2} \left( 1 + \frac{1}{45} s^2 + \ldots + \frac{1}{2} s(1 - \frac{1}{3} s) + \ldots \right) .
\]

- Heat kernel for the full vector multiplet including fermions:

\[
K_V^V(s) = \frac{1}{4\pi^2 a^4} \left( \frac{1}{2s} - \frac{1}{12} + \ldots \right) .
\]

- A $1/s$ term was generated by interactions.
- The constant term changed sign due to interactions.
The Hyper-Multiplet

- The bosons in the hyper multiplet are canonical – just four free fields.

- The fermions interact with the graviphoton so the conformal weights differ from a free field.

- The $S^2$ tower of fermions is shifted relative to a free fermion.

- Heat kernel for the complete hyper-multiplet:

\[ K^H_4(s) = \frac{1}{4\pi^2 a^4} \left( -\frac{1}{s} - \frac{1}{12} + \ldots \right). \]

- A $1/s$ term was generated by interactions.

- The constant term changed sign due to interactions.
The Vector-Multiplet: Boundary

- The vector multiplet has a feature not yet discussed: gauge invariance.

- Two auxiliary towers cancel: unphysical states (violate gauge condition) and physical (but pure gauge).

- The boundary state: one of the would-be gauge functions is not normalizable so one state survives.

- Alternatively: one equation of motion is a constraint so one spatial profile survives.

- The boundary state is a massless boson on $S^2$:

$$-\nabla^I \delta A_I = -\nabla^2 \Lambda = 0$$

- Final result for the heat kernel:

$$K_4^V(s) = \frac{1}{4\pi^2 a^4} \left( \frac{1}{2s} - \frac{1}{12} \right) + \frac{1}{4\pi^2 a^4} \left( \frac{1}{2s} + \frac{1}{6} \right) = \frac{1}{4\pi^2 a^4} \left( \frac{1}{s} + \frac{1}{12} \right) .$$
The (Massive) Gravitino Multiplet

- Bulk modes: bosons and fermions all have conformal weight shifted from the free value.
- Boundary modes: two vectors each have a gauge symmetry and so a boundary scalar.
- The SUSY variation is a fermionic gauge symmetry of the gravitino that gives a boundary fermion
  \[ \gamma^I \nabla_I \epsilon = 0 \).
- The boundary heat kernel is constant because of boson-fermion degeneracy
  \[ K^{(3/2)}_{\text{bndy}} = \frac{1}{4\pi^2 a^4} \cdot \frac{1}{2} \).
- The full heat kernel:
  \[ K^{(3/2)} = \frac{1}{4\pi^2 a^4} \cdot \left( \left( -\frac{1}{s} + \frac{5}{12} \right) + \frac{1}{2} \right) = \frac{1}{4\pi^2 a^4} \cdot \left( -\frac{1}{s} + \frac{11}{12} \right) \]
**The Graviton Multiplet**

- **Five bosonic boundary modes**: four from diffeomorphisms and one from gauge symmetry.

- Boundary modes for diffeomorphisms **acquire a mass**
  \[
  (g_{IJ} \nabla^2 + R_{IJ}) \xi^J = 0 .
  \]

- The $S^2$ vectors have helicity $\lambda = \pm 1$ so angular momenta $j = 1, 2, \ldots$

- The mass of modes due to $S^2$ diffeomorphisms
  \[
  m^2 = k(k + 1) - 2 ; \quad k = 1, 2, \ldots
  \]

- The mass of modes due to AdS$_2$ diffeomorphisms
  \[
  m^2 = k(k + 1) + 2 , \quad m^2 = k(k + 1) ; \quad k = 0, 1, 2, \ldots
  \]
• **Four fermionic boundary modes** (two preserved SUSYs) with contribution to mass from background graviphoton

\[ m^2 = (k + 1)^2 - 1, \quad k = 0, 1, \ldots \]

• The heat kernel for all **boundary modes** in the graviton multiplet

\[
K_{\text{grav bndy}} = \frac{1}{4\pi^2 a^4} \cdot \frac{5}{2} \left( \frac{1}{s} + \frac{1}{3} \right) - \frac{1}{4\pi^2 a^4} \left( \frac{2}{s} + \frac{5}{3} \right) = \frac{1}{4\pi^2 a^4} \left( \frac{1}{2s} - \frac{5}{6} \right)
\]

• **Bulk modes**: bosons and fermions all have conformal weight shifted from the free value.

• Full heat kernel

\[
K_{\text{grav}} = \frac{1}{4\pi^2 a^4} \left( \left( \frac{1}{2s} - \frac{1}{12} \right) + \left( \frac{1}{2s} - \frac{5}{6} \right) \right) = \frac{1}{4\pi^2 a^4} \left( \frac{1}{s} - \frac{11}{12} \right)
\]
The Quadratic Divergence

- Heat kernel for all multiplets, including physical states in bulk and on boundary
  \[ K_{\text{phys}} = \frac{1}{4\pi^2a^4} \left[ \left( \frac{1}{s} - \frac{11}{12} \right) + (\mathcal{N} - 2) \cdot \left( -\frac{1}{s} + \frac{11}{12} \right) + n_V \left( \frac{1}{s} + \frac{1}{12} \right) + n_H \left( -\frac{1}{s} - \frac{1}{12} \right) \right] \]

- Contributions to the quadratic divergence (the 1/s term): interactions in bulk and counting boundary degrees of freedom.

- Net result: alternating sign.

- Special case \( \mathcal{N} \geq 4 \) theory (with any matter): quadratic divergence cancels (a consistency check).

- For \( \mathcal{N} = 3 \): all divergences cancel for any \( n_V = n_H \).

- For \( \mathcal{N} = 2 \): a new result.
4D Zero Modes: General

• 4D zero modes: AdS$_2$ *boundary states and also massless* on $S^2$.

• Physical origin: the *global part* of each unbroken gauge symmetry.

• Zero-modes play a special role in the 4D heat kernel:

$$D(s) = \sum_i e^{-s\lambda_i} = \sum_{\lambda_i \neq 0} e^{-s\lambda_i} + N_0$$

• The path integral reduces to an *ordinary* integral

$$e^{-W} = \int \mathcal{D}\phi_0 = \text{Vol}[\phi_0] \sim \epsilon^{-N_0\Delta}.$$ 

• The correct zero-mode contribution: *larger than the naïve result by a factor of the scaling dimension* $\Delta$. 
4D Zero Modes: Computation

• Vector fields: no new issue since $\Delta = 1$ for a vector field.

• Bosonic 0-modes in SUGRA multiplet: 6 diff’s on $S^2$ (two with $j = 1$) and scaling dimension $\Delta_2 = 2$. (Heat kernel counts as if $\Delta_2 = 1$).

• Fermionic 0-modes in SUGRA multiplet: 8 preserved SUSYs $\Delta_{3/2} = \frac{3}{2}$. (Heat kernel counts as if $\Delta_{3/2} = \frac{1}{2}$).

• Correction due to 0-modes

$$K_{zm} = \frac{1}{8\pi^2 a^4} \cdot \left[ 6 \cdot (2 - 1) - 8 \cdot \left( \frac{3}{2} - \frac{1}{2} \right) \right] = \frac{1}{4\pi^2 a^4} (-1) .$$

• Note: *much of the literature accounts incorrectly for 0-modes.*
Example: Reissner-Nordström

Consider a purely bosonic solution: gravity+Maxwell.

Contributions are the bosonic terms from the $\mathcal{N} = 2$ SUGRA multiplet:

- Four free bulk bosons (2 gravity + 2 gauge field): $\delta S = -\frac{1}{45} \log A_H$.
- Interactions (bulk bosons not quite free): $\delta S = -\frac{3}{2} \log A_H$.
- 5 Boundary modes (4 gravity+1 gauge field): $\delta S = -\frac{5}{6} \log A_H$.
- Zero-modes: $\delta S = -3 \log A_H$.

Total: $\delta S = -\frac{241}{45} \log A_H$.

(Fermions in SUGRA multiplet add $\delta S = \frac{1309}{180} \log A_H$)
Summary

We re-computed quadratic fluctuation determinants around an AdS$_2 \times S^2$ near horizon geometry.

Some features of our strategy:

- Setting: a general theory with $\mathcal{N} \geq 2$ SUSY.
- Focus on states that are on-shell.
- Interactions due to background: encoded in chiral primaries.
- Compute also the renormalization of the gravitational coupling constant (quadratic divergence, $1/s$ term in the heat kernel).
- Contributions from bulk (4D), Boundary (2D), and Zero-mode (0D).