Unravelling the String Dual of the Gaussian Matrix Model

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Outline of the talk

- The Background
  - What we know about Matrix Models.
- Belyi Maps
  - Interpreting matrix integrals as a sum over special holomorphic maps.
- Worldsheets from Feynman Graphs
  - A worldsheet construction and integer Strebel Differentials.
- The Skeleton of Gauge-String Duality
  - An AdS/CFT picture for correlation functions.
- Final Remarks

Based on

Work in Progress

Worldsheet Construction:

- R. Gopakumar (2004-06).
- S. Razamat (arXiv: 0803.2681, 0911.0658).

Matrix Integrals and Belyi Maps:

- R. de Mello Koch and S. Ramgoolam (arXiv: 1002.1634).

1 The Background

The Background

- The AdS/CFT correspondence has changed the nature as well as reshaped the focus of string theory.
- After more than a decade the connection between gauge theory and gravity (string theory) still appears as miraculous and magical as ever.
- Physicists are not supposed to believe in magic or miracles - our task is to demystify!
- We need to lay bare the nuts and bolts of this correspondence and understand what makes it tick without invoking string dualities.
- That would enable us to see how general the correspondence is and understand more examples and apply it to newer settings.
- Need to start with a simple enough toy model which is both tractable as well as exhibits the necessary complexity.

The Background

- ✦ Study the one hermitian matrix model

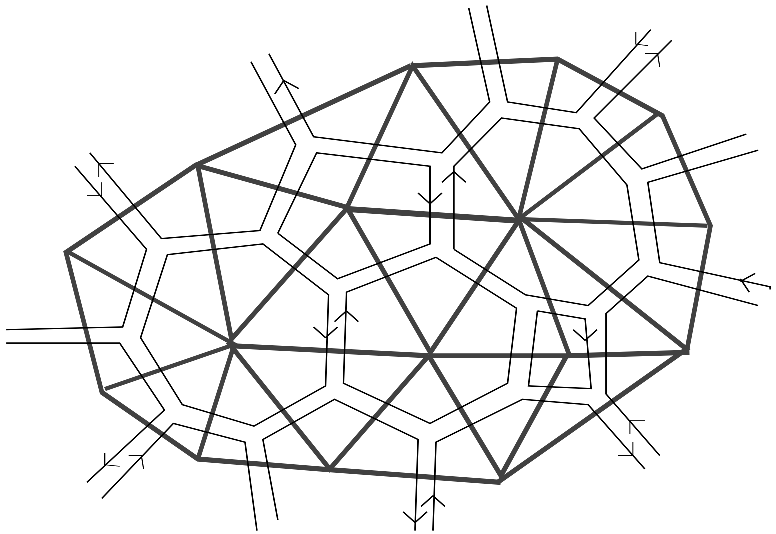
$$Z[t_k] = \int [dM]_{N\times N} e^{-N\mathrm{Tr}M^2 + \sum_k t_k N\mathrm{Tr}M^{2k}}$$

.

- ✦ To be thought of as the generating function of correlators $\langle \prod_i \mathrm{Tr}M^{2k_i} \rangle$ in the gaussian ("free") matrix integral.
- ✦ How do the Feynman-'t Hooft Diagrams of this theory (in the 't Hooft limit) generate a closed string dual?
- ✦ Can hope to understand with precision (in an AdS/CFT like formulation) since this would exhibit the structural bare bones of the gauge-string duality.
- ✦ And provide a starting point to deciphering the dual to free gauge theory.
- ✦ Moreover, Gaussian matrix integrals arise as consistent subsectors within gauge theories (e.g. Supersymmetric Wilson Loops in $\mathcal{N} = 4$ Yang-Mills theory).

The Background *continued*

- In '80-90's, this model was studied in the "double scaling limit".
- Amplitudes dominated by Feynman diagrams with many vertices - dual graph has many regular polygonal faces - dense "triangulations" of a Riemann surface.



- Intuition: The matrix integral in this limit approximates the continuum sum over the moduli space of Riemann Surfaces.
- Dual to non-critical string theories. Actually minimal Topological string theories of topological 2d gravity with its set of physical operators $\mathcal{O}_k$.

The Background *continued*

- Here we are interested in the conventional 't Hooft limit (N large but finite, t_k finite).

- Earlier Intuition: "Non-universal" theory describing a discretely triangulated world-sheet. No continuum string theory description.

- At odds with the seeming universality of gauge-string correspondence.

- In fact, 'tHooft dual believed to be B-model topological string theory on deformed conifold (Dijkgraaf-Vafa).

- What is wrong with the triangulation intuition?

Replace view of discrete worldsheet by the notion of amplitudes *localised* at discrete set of points on the moduli space of Riemann surfaces.

2 Belyi Maps

Belyi Maps

- Interesting observation of de Mello Koch and Ramgoolam on Belyi maps.
- Can interpret the contributions to the correlators $\langle \prod_i^n \mathrm{Tr} M^{2k_i} \rangle$ as a sum over particular holomorphic maps $X : \Sigma_{g,n} \to \mathbb{P}^1$.
- Basically, the different Wick contractions of the Gaussian model can be organised in terms of three sets of elements of the permutation group S_{2k} where $2k = \sum_i 2k_i$.
- The first one $\beta \in (2k_1)(2k_2)\ldots(2k_n)$ captures the cyclic information at each vertex of the n-point function.
- The second one $\alpha \in [2^k]$ consists only of two cycles (transpositions) and captures the various possible Wick contractions (edge information).
- The power of N for a given Wick contraction (number of faces) is associated with the number of disjoint cycles C_γ in the permutation $\alpha \cdot \beta \equiv \gamma^{-1}$.

Belyi Maps *continued*

- These permutations can be naturally associated with branching structure (ramification profile) of holomorphic maps from Riemann surfaces to Riemann surfaces (Riemann).
- Thus $N^{n-k}\langle\prod_i^n \mathrm{Tr} M^{2k_i}\rangle_{conn} = \sum_{g=0}\sum_{\alpha\in[2^k]} N^{2-2g}$.
- Where $(2-2g) = C_\gamma + C_\beta - k = 2k(2-2\times 0) - B$.
- With $B = (2k - C_\gamma) + (2k - C_\beta) + (2k - C_\alpha)$.
- This is the Riemann-Hurwitz formula for holomorphic maps from Σ_g to $\mathbb{P}^1$ where $2k$ is the degree of the map and B is the branching number.
- In this case the holomorphic map is branched over three points (e.g. $0, 1, \infty$) with ramification profiles given by the permutations α, β, γ (with $\alpha\cdot\beta\cdot\gamma = 1$).
- Thus matrix correlators can be interpreted as *counting* Holomorphic maps from $\Sigma_{g,n} \to \mathbb{P}^1$ which happen to be branched over three points (with "simple" branching over one of the points). (R. dMK and S.R)

Belyi Maps *continued*

- These are precisely the class of maps known to mathematicians as (clean) Belyi maps.

- Significant because not every Riemann Surface admits a Belyi map - only a discrete set of points on moduli space do so.

- These are the *arithmetic* riemann surfaces - those defined over the algebraic numbers $\bar{\mathbb{Q}}$ (Belyi).

- Related to Grothendieck's program (*Dessins L'Enfants*) to understand the representations of the absolute Galois group - $Gal(\bar{\mathbb{Q}}/\mathbb{Q})$.

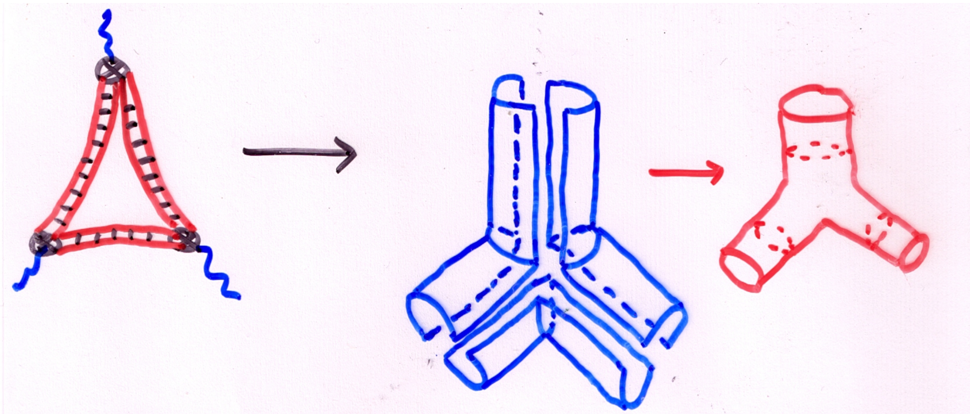
- But can we have a continuum string amplitude pick out a set of discrete points on moduli space?

- Can we write an explicit worldsheet description of such a string theory?

3 Worldsheets from Feynman Graphs

Worldsheets from Feynman Graphs

- A general approach to understanding gauge-string duality is to associate world-sheets to individual Feynman diagrams in a very natural way. (R.G.)
- Glue together semi-infinite cylinders along the faces of the dual Feynman graph - vertices of the original Feynman graph.



- *All* inequivalent Riemann Surfaces can be generated this way. They are parametrised by the (Strebel) lengths of the edges of the dual graph.
- Strebel differential $\phi(z)dz^2 \rightarrow$ An almost flat worldsheet metric ("Strebel Gauge") $ds^2 = |\phi(z)|dzd\bar{z}$. Vertices of the dual graph are zeroes of $\phi(z)$.
- Proposal is to identify Strebel lengths with (inverse) Schwinger proper times.

Worldsheets from Feynman Graphs *continued*

- For the case of matrix integrals (where there is no notion of proper time) a modification proposed by Razamat.
- Identify Strebel length (of the edge of the dual graph) with the (integer) *number of contractions* along the original edge. Equivalently, face dual to the vertex with TrM^k has length k.
- Exactly the same lengths (recall regular polygons) as associated in triangulation approach. But interpretation is different.
- The associated Riemann surfaces are continuous (formed piecewise cylindrically) and not discrete.
- But integer values of Strebel Lengths pick out a set of discrete points (dense) on the moduli space of Riemann surfaces.
- Let us now put together these inputs into a coherent picture of how the string worldsheet as well as target space are built up from the large N 'tHooft diagrams of the matrix integral.

4 The Skeleton of Gauge-String Duality

The Skeleton

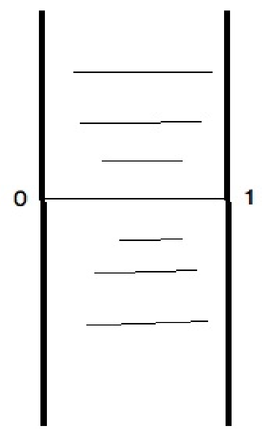
- How can a well defined continuum string theory be defined only on a discrete set of points on moduli space?
- A simple example: Consider the A-model topological string theory with T^2 as its target.
- The sigma model path integral gets contributions only from holomorphic maps from the worldsheet to the target.
- At genus one, for fixed complex structure τ_{ST} of the target T^2, such maps exist only for special values of the worldsheet τ_{WS}.
- τ_{WS} must be related to τ_{ST} by special $GL(2, \mathbb{Z})$ transformations. Picks out a discrete set of points in the fundamental domain of τ_{WS}.
- Therefore the string ampltude gets its contribution from a sum of delta functions on the moduli space. Belyi maps suggest something similar in this case.

The Skeleton

- Can use a very explicit description of Belyi maps (Mulase and Penkava).
- Shows that Belyi maps are admissible precisely when the Strebel lengths are integers!
- Thus the same discrete set of points on moduli space are picked out as in the world-sheet construction.
- Natural to view the string dual for the matrix model as getting localised contributions from discrete points in moduli space corresponding to integer Strebel lengths.
- This "discretisation" of moduli space can nevertheless capture the topological information of intersection numbers on moduli space (and more). (Chekhov, Makeenko)
- Topological A-model String theory provides the means for such a localisation on moduli space through holomorphic maps - Belyi maps, in this case.

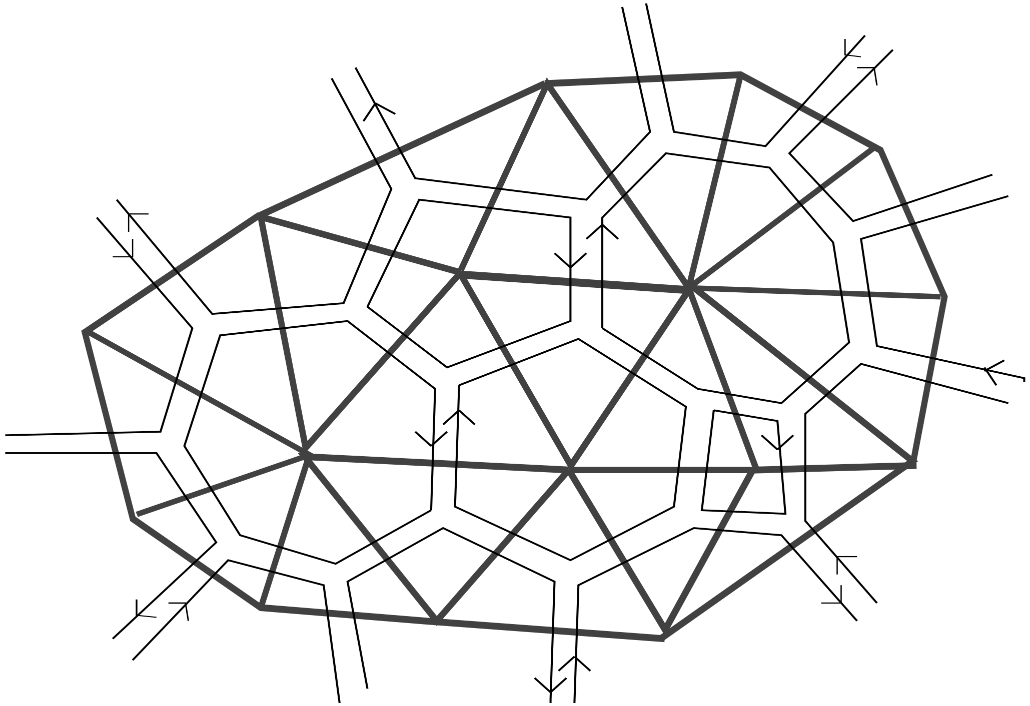
The Skeleton *continued*

- Worldsheet riemann surface is built out of gluing vertical strips in the complex plane each of width equal unit Strebel Length ($0 < Re(z) < 1$).
- Essentially these are the fattened 'tHooft lines of the original graph which arise in the Wick contractions between any two vertices (insertions of TrM^{2k_i}).



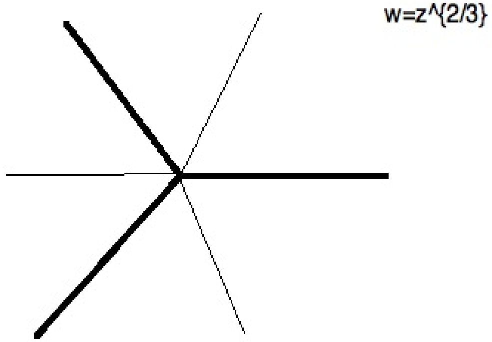
- The horizontal interval $[0, 1]$ is therefore an edge of the dual graph.
- The Strebel differential is simply dz^2 in each such strip.

The Skeleton *continued*

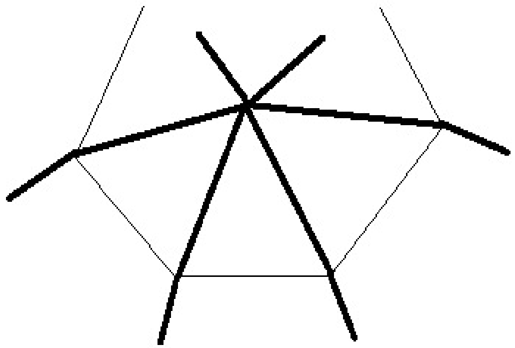


The Skeleton *continued*

- At each dual vertex of order p (i.e p-gonal face of original graph), e.g. $z = 0$, the gluing is through $w = z^{\frac{2}{p}}$ ("closing up holes").



- At the centre of each k-gonal dual face (i.e. k-fold vertex of the original graph), e.g. $z = i\infty$, the gluing is through $u \propto e^{\frac{2\pi i z}{k}}$.



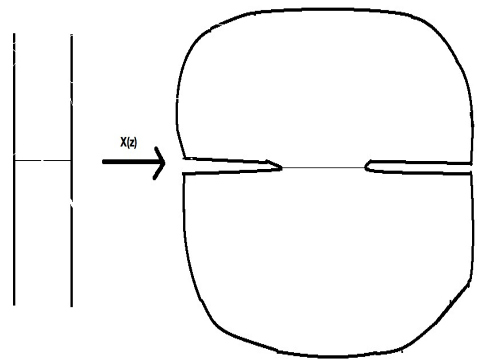
The Skeleton *continued*

- The various Feynman diagrams for the n-pt. function are supposed to be in 1-1 correspondence with *clean* Belyi maps (of degree $2k$) from this glued up worldsheet.

- We can now explicitly see this: holomorphic Belyi map on each strip

$$X(z) = \sin^2 \frac{\pi z}{2}.$$

Each strip of width one is mapped once onto a $\mathbb{P}^1$.



- Essentially the vertices of the dual graph ($z = 0, 1$ on the strip) are mapped to 0 and 1 and the original vertices ($z = \infty$) to ∞, the branch points in the target space.

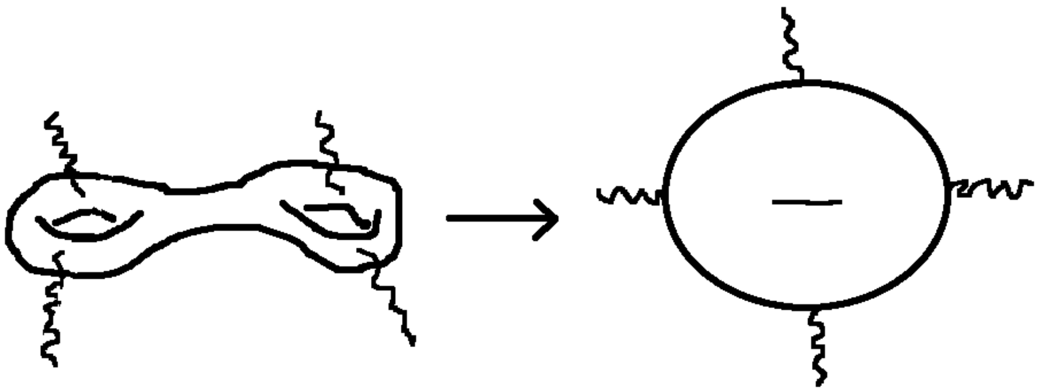
The Skeleton *continued*

- k separate strips in any Feynman diagram $\rightarrow$ cover of $\mathbb{P}^1$ k times.
- What is the interpretation of this Belyi map?
- The differential $dz^2 = \frac{dX^2}{\pi^2 X(1-X)} = \frac{1}{\pi^2}(\frac{dX}{y})^2 \equiv \omega^2$ where $y^2 = X(1-X)$.
- $\omega = \frac{1}{\pi}\frac{dX}{y}$ is the canonical abelian differential on $\mathbb{P}^1$ which is a double cover of the X plane. Hence the nett map is of degree $2k$.
- We thus see that the target space $\mathbb{P}^1$ is nothing other than that of the complexified eigenvalues (single cut) of the Gaussian model!
- The spatial slices of the worldsheet (the interval $[0,1]$ on each strip) mapped onto the eigenvalue cut $[0,1]$ (the "emergent space" of the target).

The Skeleton *continued*

- External Vertex operator insertions $\mathcal{O}_{k_i} \leftrightarrow TrM^{2k_i}$ at infinity ("UV") have a specific branching number k_i.
- Different Feynman graphs $\rightarrow$ sum over holomorphic maps with fixed branching data at infinity: $(k_1)\ldots(k_n)$.
- Leads to additional branching at 0 and 1 ("IR"), the two endpoints of the holographically emergent "space".
- Basically the strings coming in from infinity split and join at these specific points (cf. Gross-Mende, Eynard-Orantin).
- These holomorphic maps *are* stringy Witten diagrams in the target space $\mathbb{P}^1$.

The Skeleton *continued*



- The full target space ought to be a non-compact CY 3-fold for consistency.
- A topological A-model mirror description of Dijkgraaf-Vafa duality?
- A $\mathbb{P}^1$ resolution of $xy + (e^u - 1)(e^v - 1) = 0$?
- However, free Gaussian likely to be somewhat singular.
- Need to understand how the full CY target space geometry emerges from the matrix integral. A bit like understanding the S^5 of the $AdS_5 \times S^5$.

To Conclude....To Continue

- Matrix models give a hands on picture of gauge-string duality with some unusual features.
- A dual topological string with amplitudes localised at special points on the moduli space.
- Amplitudes count holomorphic maps to $\mathbb{P}^1$ with three branchpoints.
- The Feynman diagrams exhibit this string theory in a (singular) "Strebel gauge" for the metric.
- In this gauge easy to reconstruct (part of) the target space - Riemann surface associated with the matrix model eigenvalues.
- The holomorphic (Belyi) maps describe the stringy Witten diagrams in the target space $\mathbb{P}^1$.
- Can hope to draw lessons for the string duals of free (super) Yang-Mills which one expects to be quasi-topological.

The end