Gravitational duality: a NUT story

Francois Dehouck

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"Supersymmetry and gravitational duality": R. Argurio, L. Houart, F.D. [PRD 79:125001, 2009]
"Boosting Taub-NUT to a BPS NUT-wave": R. Argurio, L. Houart, F.D. [JHEP 0901:045,2009]
"Why not a di-NUT ? ": R. Argurio, F.D. [hep-th:0909.0542]

Understanding Quantum Gravity...

One of the goals of theoretical physics is to find a quantum theory for gravity...

...through S-duality

Duality between weakly and strongly coupled sectors of a theory is a powerful tool to delve into its non-perturbative physics. Supersymmetry helps in providing protected quantities that can be compared in both weakly and strongly coupled (or electric and magnetic) sectors.

Goal of this talk

- presence of dyonic metrics in general relativity and an adapted EM duality in linearized Gravity.
- Show the presence of this duality in supergravity. Establish the supersymmetry of duality rotated supersymmetric solutions.

Outline

1. Electromagnetic duality
2. Duality in Linearized Gravity + Examples
3. $\mathcal{N} = 2$ Supersymmetric solutions with NUT charge
4. $\mathcal{N} = 1$ Supersymmetric solutions with NUT charge
5. Conclusions and future work

Duality in EM

Duality in electromagnetism states that for every "electric" field strength, there is a dual "magnetic" field strength. The duality is a Hodge duality:

$$\begin{aligned} F^{\mu\nu} &\rightarrow \tilde{F}^{\mu\nu} \equiv (*F)^{\mu\nu} = \frac{1}{2}\varepsilon^{\mu\nu\rho\sigma}F_{\rho\sigma} \\ Q &\rightarrow H \end{aligned}$$

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Example: Coulomb charge vs. magnetic monopole

$$A = \frac{Q}{r}\, dt \qquad F = \frac{Q}{r^2}\, dt \wedge dr$$

$$\tilde{F} = H\, sin\theta\, d\theta \wedge d\phi \qquad \tilde{A} = -H\, cos\theta\, d\phi$$

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$$\tilde{F} = H\, sin\theta\, d\theta \wedge d\phi \qquad \tilde{A} = -H\, cos\theta\, d\phi = -H\, \frac{z}{r(r^2 - z^2)}(xdy - ydx)$$

Note: If we look at the gauge potential, the magnetic monopole has a Dirac string singularity along the z-axis.

The monopole as a magnetic charge H:

Introduce a magnetic current in the Bianchi identity:

$$d * F = 4\pi J_{el}$$
$$dF = 4\pi J_{magn}$$

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Is there something similar in linearized gravity ?

Duality in linearized gravity

The Lorentzian Taub-NUT solution found in [Taub '51; Newman, Tamburino, Unti '63] is:

$$ds^2 = -\frac{\lambda^2}{R^2}[dt + 2Ncos\theta d\phi]^2 + \frac{R^2}{\lambda^2}dr^2 + R^2[d\theta^2 + sin^2\theta d\phi^2]$$

where $\lambda^2 = r^2 - 2Mr - N^2$ and $R^2 = r^2 + N^2$

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The duality can be expressed as a Hodge duality on the Riemann tensor:

$$R_{\mu\nu\rho\sigma} \rightarrow \tilde{R}_{\mu\nu\rho\sigma} = \frac{1}{2}\,\varepsilon_{\mu\nu\alpha\beta}R^{\alpha\beta}{}_{\rho\sigma} \qquad M \rightarrow N$$

[Henneaux, Teitelboim '04]

Generalizing the EM idea: [Bunster, Cnockaert, Henneaux, Portugues '06]

The duality is taken on the Lorentz indices: $\tilde{R}_{\mu\nu\rho\sigma} = \frac{1}{2}\,\varepsilon_{\mu\nu\alpha\beta}R^{\alpha\beta}{}_{\rho\sigma}$.

We introduce a "magnetic" stress-energy tensor $\Theta_{\mu\nu}$:

$$
\begin{aligned}
&G_{\mu\nu} = 8\pi\, T_{\mu\nu} \\
&R_{\mu\nu\alpha\beta} + R_{\mu\beta\nu\alpha} + R_{\mu\alpha\beta\nu} = 0 \\
&\partial_\epsilon\, R_{\gamma\delta\alpha\beta} + \partial_\alpha\, R_{\gamma\delta\beta\epsilon} + \partial_\beta\, R_{\gamma\delta\epsilon\alpha} = 0
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Solution: $R_{\alpha\beta\lambda\mu} = r_{\alpha\beta\lambda\mu} + f(\Phi) \;\rightarrow\; \partial_\alpha \Phi^{\alpha\beta}{}_{\gamma} = -16\pi\Theta^{\beta}{}_{\gamma}$

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Charges in general relativity:

$$
P_\mu = \int T_{0\mu}\, d^3x \qquad L^{\mu\nu} = \int (x^\mu T^{0\nu} - x^\nu T^{0\mu}) d^3x
$$

$$
K_\mu = \int \Theta_{0\mu}\, d^3x \qquad \tilde{L}^{\mu\nu} = \int (x^\mu \Theta^{0\nu} - x^\nu \Theta^{0\mu}) d^3x
$$

[Ramaswamy, Sen '81],[Ashtekar, Sen '82],[Mueller, Perry '86], [Bossard, Nicolai, Stelle '09]

1. The Kerr-NUT black hole [Carter '68] :

$$ds^2 = -\frac{\lambda^2}{R^2}[dt - (asin^2\theta - 2Ncos\theta)d\phi]^2 + \frac{sin^2\theta}{R^2}[(r^2 + a^2 + N^2)d\phi - adt]^2 + \frac{R^2}{\lambda^2}dr^2 + R^2 d\theta^2,$$

where $\lambda^2 = r^2 - 2Mr + a^2 - N^2$ and $R^2 = r^2 + (N + acos\theta)^2$

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- Linearized Schwarzschild ($a = 0, N = 0$): $P_0 = M$
- Linearized NUT ($a = 0, M = 0$):
  - ${\Phi^{0z}}_0 = -16\pi N\delta(x)\delta(y)\vartheta(z) \Rightarrow \Theta^{00} = N\delta(\mathbf{x}) \Rightarrow K_0 = N.$

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  - $\Phi_{\mu\nu\rho} = 0 \Rightarrow \Delta L^{xy}/\Delta z = N$ [Bonnor '69]

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  - $\Phi^{0z}{}_0 = -16\pi N\delta(x)\delta(y)\vartheta(z) \qquad \Phi^{0y}{}_x = -\Phi^{0x}{}_y = -\Phi^{xy}{}_0 = 8\pi Na\delta(\mathbf{x})$
  - This describes a magnetic mass $K_0 = N$ with a dual angular momentum $\tilde{L}^{xy} = Na$.

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- "exotic" interpretations: [Argurio, F.D. '09]
  - "Physical" Kerr with $\Phi^{z0}{}_0 = 16\pi Ma\delta(\mathbf{x})$
  - $T_{00} = M\delta(\mathbf{x}) \qquad \Theta_{00} = Ma\delta(x)\delta(y)\delta'(z)$
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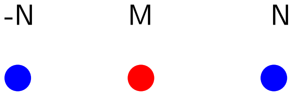

In the limit where
$\epsilon \longrightarrow 0,\ N \longrightarrow \infty$
and $N\epsilon = Ma$

2. The metric of the shock pp-wave is:

$$ds^2 = H(x, y, u)du^2 - dudv + dx^2 + dy^2$$

where $H(x, y, u) = V(x, y)\delta(u)$ and V is harmonic in x and y.

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Example: Aichelburg-Sexl pp-wave vs. NUT-wave

- The boosted Schwarzschild with $\gamma \rightarrow \infty$, $M \rightarrow 0$ and $M\gamma = p$:

$V(x, y) = -8\, p\, ln(\sqrt{x^2 + y^2})$ Charges: $P_0 = p = |P_3|$

[Aichelburg, Sexl '71],[Dray, t'Hooft '85]

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  [Aichelburg, Sexl '71],[Dray, t'Hooft '85]

- The boosted NUT metric with $\gamma \rightarrow \infty$, $N \rightarrow 0$ and $N\gamma = k$

  $\tilde{V}(x, y) = -8\,k\,arctan(x/y)$ Charges: $K_0 = k = |K_3|$

[Argurio, F.D., L. Houart '09]

$\mathcal{N} = 2$ Supersymmetric solutions with NUT charge

We consider $\mathcal{N} = 2$ pure supergravity in D=4

gravity multiplet: $g_{\mu\nu}, \psi_\mu, A_\mu$

The charged Taub-NUT solution [Brill '64] is a solution of the bosonic e.o.m.:

$$ds^2 = -\frac{\lambda}{R^2}(dt + 2N\cos\theta d\phi)^2 + \frac{R^2}{\lambda}dr^2 + (r^2 + N^2)(d\theta^2 + \sin^2\theta d\phi^2)$$

where $\lambda = r^2 - N^2 - 2Mr + Q^2 + H^2$ and $R^2 = r^2 + N^2$

$$A_t = \frac{Qr + NH}{r^2 + N^2} \qquad A_\phi = \frac{-H(r^2 - N^2) + 2NQr}{r^2 + N^2}\cos\theta$$

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It is easy to see that in the case $N = 0$, we recover the supersymmetric Reissner-Nordström black hole solution.

What do we already know ?

- All supersymmetric solutions have been classified [Tod '83]
- All modifications of the (r.h.s. of the) supersymmetry algebra [Van Holten, Van Proeyen '82],[Ferrara, Porrati '98]

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Strategy

- Review of the susy of the R.N. black hole solution.
- Consider modifications of the supersymmetry algebra to deal with the supersymmetry of the charged Taub-NUT (or duals of supersymmetric solutions).

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Strategy

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Important Hint

Reissner-Nordström with Q and H can be obtained from the Q-charged solution by an EM duality rotation. [Romans '92]

To be supersymmetric, the solution must possess non-trivial killing spinors:

$$\delta\psi_\mu = \hat{\nabla}_\mu \epsilon = \hat{D}_\mu \epsilon + \frac{i}{4} F_{ab} \gamma^{ab} \gamma_\mu \epsilon = 0$$

The BPS bound for R.N. can be obtained as a necessary condition for their existence:

$$[\hat{\nabla}_\mu, \hat{\nabla}_\nu]\epsilon = 0 \ \rightarrow \ \Theta X_{\mu\nu} \epsilon = 0$$

This equation possess non-trivial solutions iff

$$det\Theta = 0 \Rightarrow \qquad M^2 = Q^2 + H^2$$

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The projection on the spinor is

$$[M - i(Q - \gamma_5 H)\gamma_0]\epsilon = 0$$

This is precisely the r.h.s. of the $\mathcal{N} = 2$ supersymmetry algebra

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This equation possess non-trivial solutions iff [Alonso-Alberca, Meessen, Ortin '00],[Kallosh, Kastor, Ortin, Torma '94],[Alvarez, Meessen, Ortin '97], [Hull '98]

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This is precisely the r.h.s. of the $\mathcal{N} = 2$ supersymmetry algebra ?

$$\{\mathcal{Q}, \mathcal{Q}^\star\} \stackrel{?}{=} \gamma^\mu CP_\mu + \gamma_5\gamma^\mu CK_\mu - i(U + \gamma_5 V)C$$

The bosonic supercharge in supergravity is

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We only consider $P_{\mu} = \lambda K_{\mu}$ where $\lambda =$cst.

- No known solutions with $P_{\mu} \neq \lambda K_{\mu}$
- r.h.s of the generalized SUSY algebra does not have vanishing eigenvalues when $P_{\mu} \neq \lambda K_{\mu}$

Hermitian superalgebra with redefined generators :

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Example: The charged Taub-NUT:

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The previous rotation is actually acting as a gravitational duality rotation on the bosonic charges: $(\alpha_m = arctan(K_0/P_0))$

$$\begin{pmatrix} \cos\alpha_m & \sin\alpha_m \\ -\sin\alpha_m & \cos\alpha_m \end{pmatrix} \begin{pmatrix} M \\ N \end{pmatrix} = \begin{pmatrix} M' \\ 0 \end{pmatrix}$$

$\mathcal{N} = 1$ Supersymmetric solutions with NUT charge

The bosonic part of the $\mathcal{N} = 1$ supergravity Lagrangian is just the Einstein-Hilbert action.
The Aichelburg-Sexl pp-wave is an half BPS solution with BPS bound

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This stems for the existence of the same phenomena in $\mathcal{N} = 1$ supergravity where the supersymmetry algebra has to be modified like:

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Conclusions:

In linearized Gravity:

- Duality invariant Einstein equations introducing a magnetic stress-energy tensor $\Theta_{\mu\nu}$.
- Dual Poincare charges: Taub-NUT, and the Kerr-NUT. New interpretation for Kerr's source.
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Future work

- Definition of the NUT charge in the full theory ? Gravitational duality for the non-linear theory ? [Compere, Virmani '09??]
- What about more supersymmetry ? ($\mathcal{N} = 4, 8$?) or higher-dimensions ? (M-theory superalgebra ?)
- Generalize these ideas to AlAdS spacetimes.
  - J enters the superalgebra (Also Witten-Nester). Dual charges ? work in progress...
  - Plebanski-Demianski solution:Λ, $Q + iH$, $M + iN$ and $a + i\alpha$: Link between rotating and C-metrics by duality ?
  - AdS/CFT: dual graviton as a source for the Cotton Tensor [Leigh, Petkou '07],[de Haro '08]

ThaNk yoU !