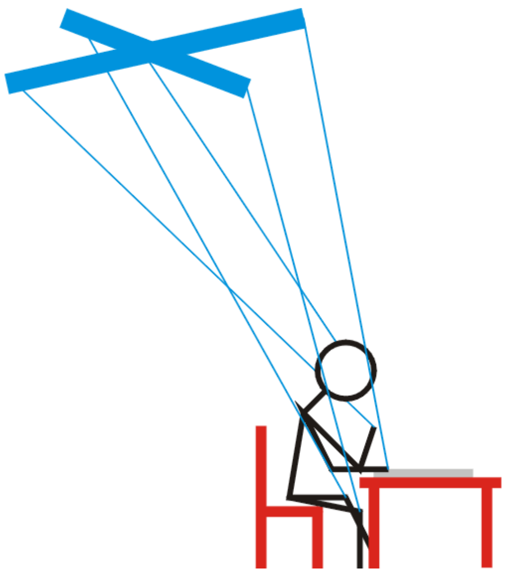
How hard is it to find their properties?
What could we do with them if we had them?



Local Hamiltonians in Quantum Computation

[QIP 2010 tutorial workshop]

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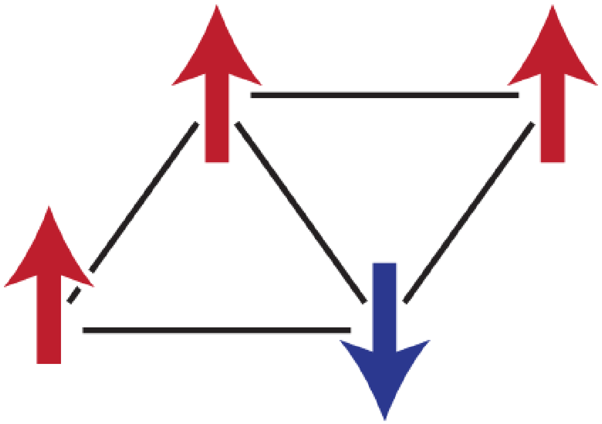
0) Intro

- Questions about the physical properties of some systems can be really hard
  - low-energy configurations
    - spin glasses
    - proteins
  - finding ground states
    … **optimization**
  - methods: simulated annealing, imaginary time evolution…
    - heuristic methods,
      average case vs. worst case
    - does quantum mechanics help?



0) **Intro: Local Hamiltonians**

- Two questions about local Hamiltonians
  - interesting (ground) state properties

    QMA-complete problems
  - continuous-time quantum computing

    BQP universality

$$H(t) = \sum_j H_j(t)$$

0) Intro: Local Hamiltonians

- Two questions about local Hamiltonians
  - interesting (ground) state properties

    QMA-complete problems

  - continuous-time quantum computing

    BQP universality

- Stronger results:
  - small locality, simple geometry
  - small energy $\times$ time cost
  - large promise/eigenvalue gaps
  - time independence, translational invariance

0) Outline

Part I (the basics)

- Computation, circuits and Hamiltonians
  - NP-completeness of SAT
  - Feynman, reversible computing, Hamiltonian quantum computers
- Finding a difficult Hamiltonian problem
  - Local Hamiltonian [Kitaev]
  - Quantum k-SAT [Bravyi]

Part II (the topics)

- Getting down to 2-local
  - projection lemma
  - perturbation gadgets
  - optional
    - Quantum 2-SAT in 1D
    - Quantum (5,3)-SAT
- Quantum computers from local Hamiltonians
  - Adiabatic quantum computation
  - quantum walks and railroad switches

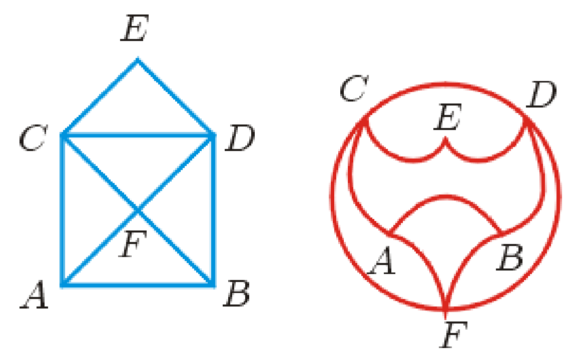
1) The class NP

- Questions (yes/no), whose answers are easy to check

- **Factoring**
  Does 114991 have
  a factor smaller than 60?

$$114991 = 1949 \times 59$$

- **Graph isomorphism**
  Are these two graphs
  isomorphic?



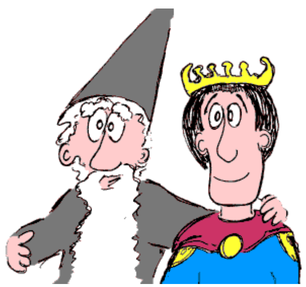
- **Satisfiability**
  Is there a bit string avoiding
  all the bad assignments?

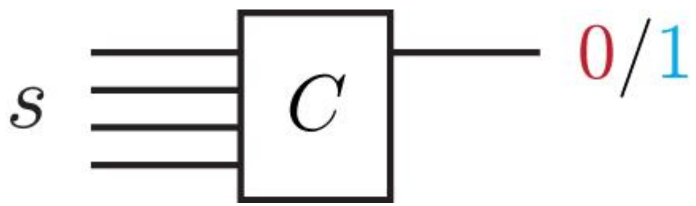
1	0	1
0	1	
0	0	
	1	1
1		0

disallowed substrings

1) The class NP

- Questions (yes/no), whose answers are easy to check
- Merlin tries to convince Arthur





a yes case: there exists a witness, on which C outputs yes

a no case: for all inputs, C outputs no

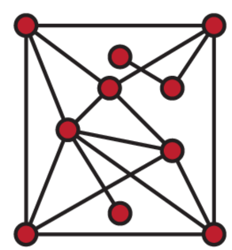
1) NP-complete problems

- Knowing how to solve a hard problem would let us solve an easier problem

does **1147** have a divisor **> 30**?

translate into an instance of a “hard” problem

is there a 4-clique?



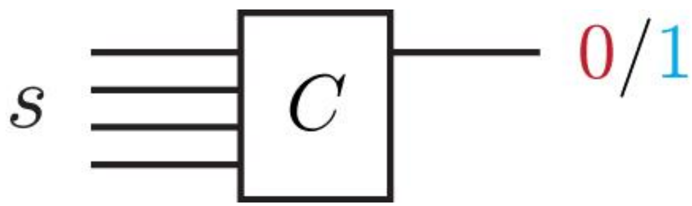
if we could do this...

YES

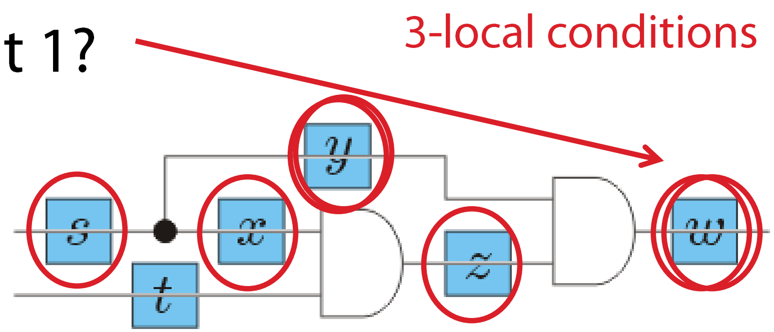
YES

1) NP-complete problems

- Knowing how to solve one NP-hard problem would let us solve all NP problems



- Could this circuit ever output 1? Does this verifier circuit have a witness?

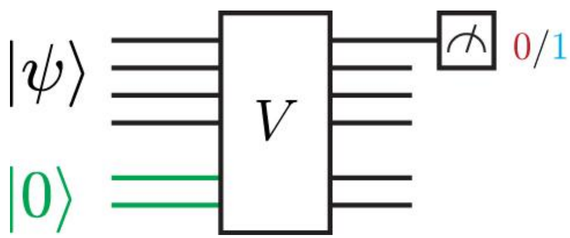


- 3-SAT is NP-complete (NP-hard, also in NP) [Cook,Levin]
- an NP-complete problem must also belong to NP

1) The class QMA

- questions (yes/no), whose answers are easy to check on a quantum computer

- Merlin tries to convince Arthur



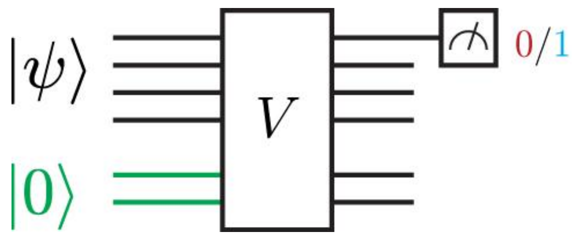
a yes case: there exists a witness, on which V outputs yes with high probability ($p \geq a$)

a no case: on any input, V outputs yes only with a small probability ($p \leq b$)

1) **Hardness for the class QMA**

- questions (yes/no), whose answers are easy to check on a quantum computer

- Merlin tries to convince Arthur



- solving a QMA hard problem = the ability to answer this:

Is there an input to this quantum verifier circuit giving output 1 with high probability?

- let's find such a problem (about Hamiltonians)

2) Implementing reversible *(quantum)* circuits

- The Schrödinger equation
  - a Hamiltonian H

$$i\frac{d}{dt}|\psi(t)\rangle = H(t)|\psi(t)\rangle$$

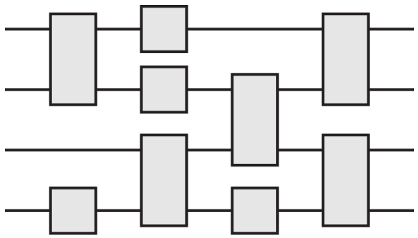
  - generates time evolution

$$|\psi(t)\rangle = U_{t,0}|\psi(0)\rangle$$

  - physical Hamiltonians: local

$$H(t) = \sum_j H_j(t)$$

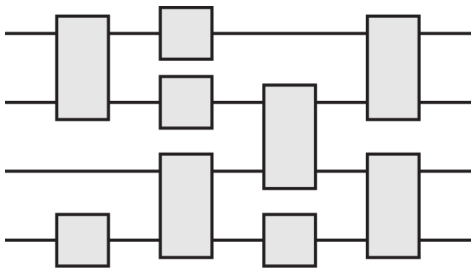
- How to make it compute?
  - a (quantum, reversible) circuit U



2) *Making a system compute: a way that won't work*

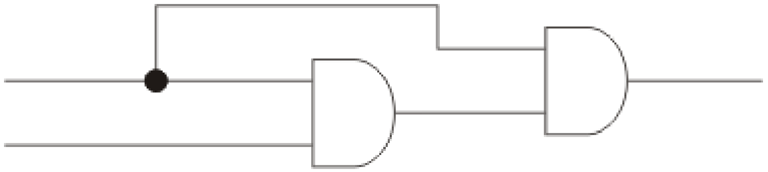
- a time evolution of a quantum system:

$$|\varphi_0\rangle\,|0\rangle\cdots|0\rangle \;\longrightarrow\; |\varphi_0\rangle\,|\varphi_1\rangle\,|\varphi_2\rangle\cdots|\varphi_L\rangle$$

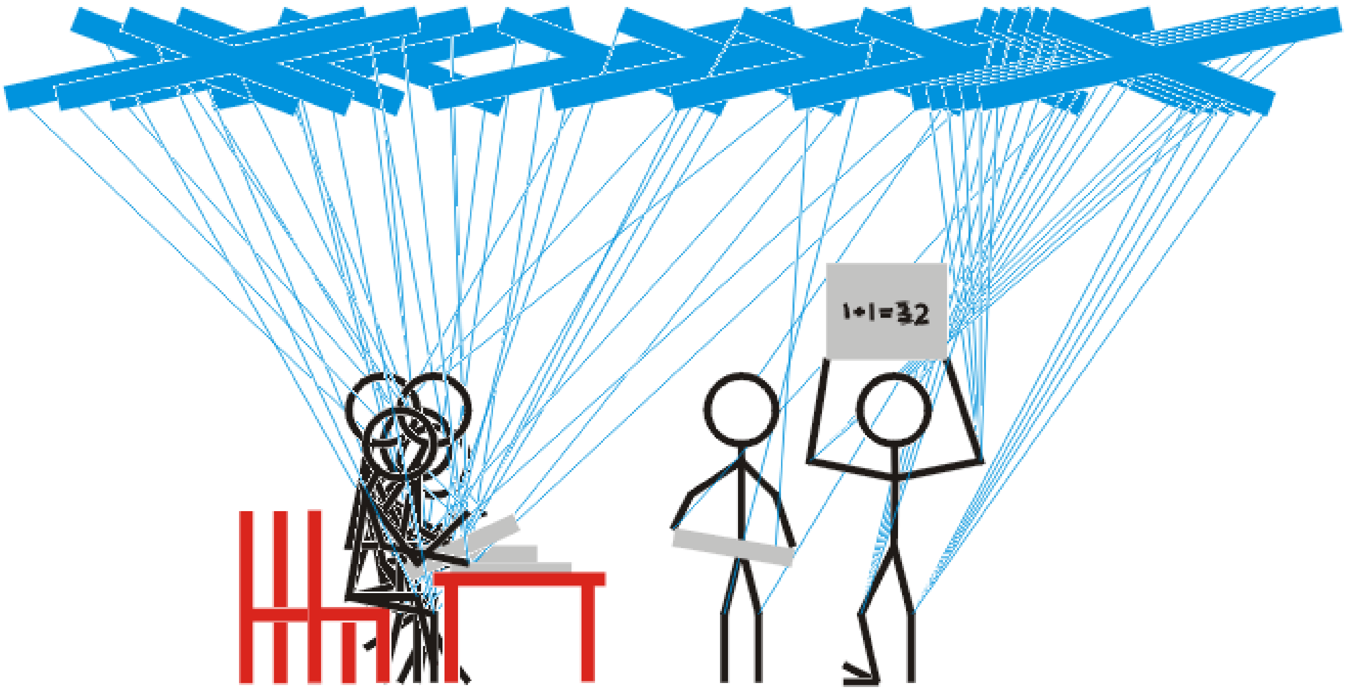


$$|\varphi_t\rangle = U_t U_{t-1}\dots U_2 U_1\,|\varphi_0\rangle$$

  - this would allow cloning!
  - classically, there's no problem with copying information...

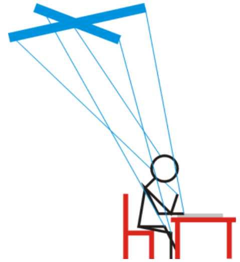


2) Feynman's Hamiltonian computer



2) **Hamiltonian quantum computation**

- Feynman's Hamiltonian computer



a pointer particle (clock register)

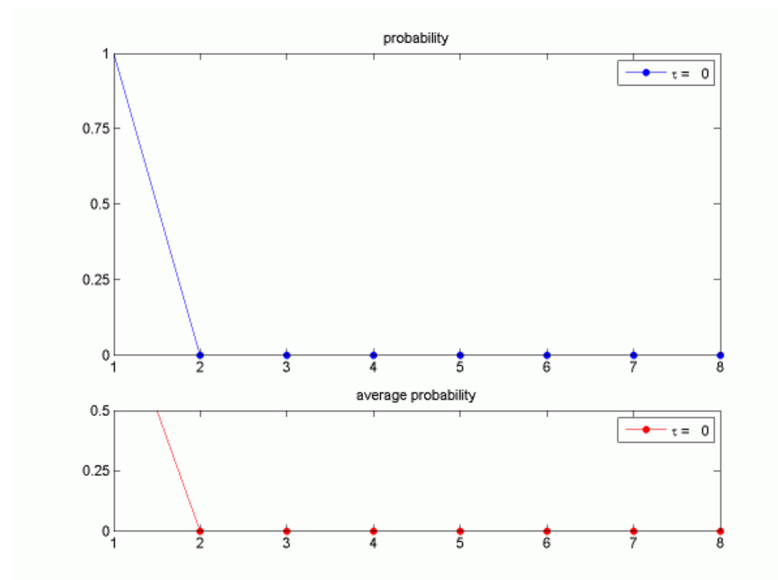
the workspace (work register)

$$|t=0\rangle_c = |10000\rangle$$
$$|t=1\rangle_c = |01000\rangle$$
$$|t=2\rangle_c = |00100\rangle$$
$$|t=3\rangle_c = |00010\rangle$$

- The Hamiltonian $$H_F = -\sum_{t=1}^{L}\left(U_t \otimes |t\rangle\langle t-1| + U_t^\dagger \otimes |t-1\rangle\langle t|\right)$$

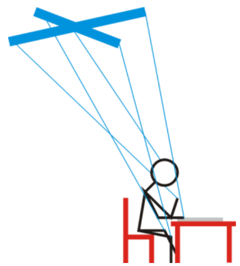
- A quantum walk on a "line"

$$|\varphi_0\rangle \otimes |0\rangle_c$$
$$U_1|\varphi_0\rangle \otimes |1\rangle_c$$
$$U_2U_1|\varphi_0\rangle \otimes |2\rangle_c$$
$$U_3U_2U_1|\varphi_0\rangle \otimes |3\rangle_c$$



2) **Hamiltonian quantum computation**

- Feynman's Hamiltonian computer



a pointer particle (clock register)

the workspace (work register)

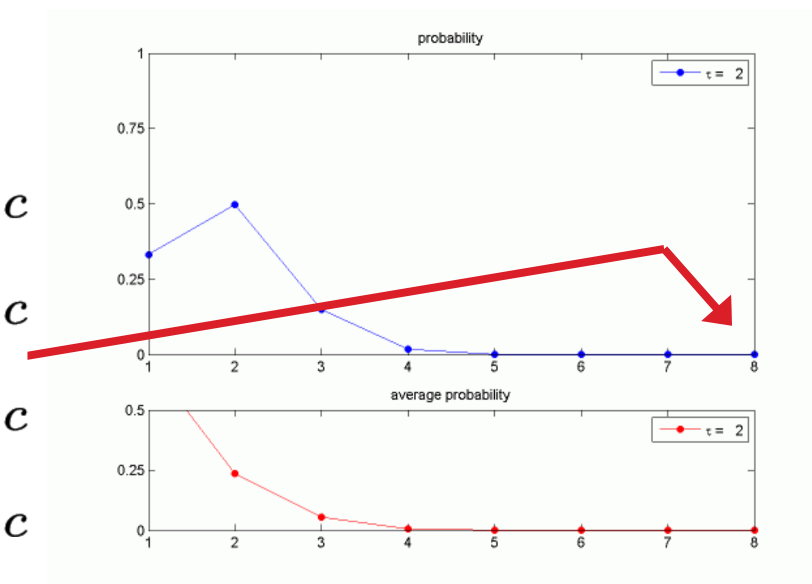
$$\begin{aligned} |t=0\rangle_c &= |10000\rangle \\ |t=1\rangle_c &= |01000\rangle \\ |t=2\rangle_c &= |00100\rangle \\ |t=3\rangle_c &= |00010\rangle \end{aligned}$$

- The Hamiltonian $H_F = -\sum_{t=1}^{L} \left(U_t \otimes |t\rangle\langle t-1| + U_t^\dagger \otimes |t-1\rangle\langle t| \right)$

- A quantum walk on a "line"

- The output

$$\begin{aligned} &|\varphi_0\rangle \otimes |0\rangle_c \\ U_1\,&|\varphi_0\rangle \otimes |1\rangle_c \\ U_2 U_1\,&|\varphi_0\rangle \otimes |2\rangle_c \\ U_3 U_2 U_1\,&|\varphi_0\rangle \otimes |3\rangle_c \end{aligned}$$



2) **Hamiltonian quantum computation**

- Feynman's Hamiltonian

$$H_F = -\sum_{t=1}^{L} \left(U_t \otimes |t\rangle \langle t-1| + U_t^\dagger \otimes |t-1\rangle \langle t| \right)$$

- The "line" of states

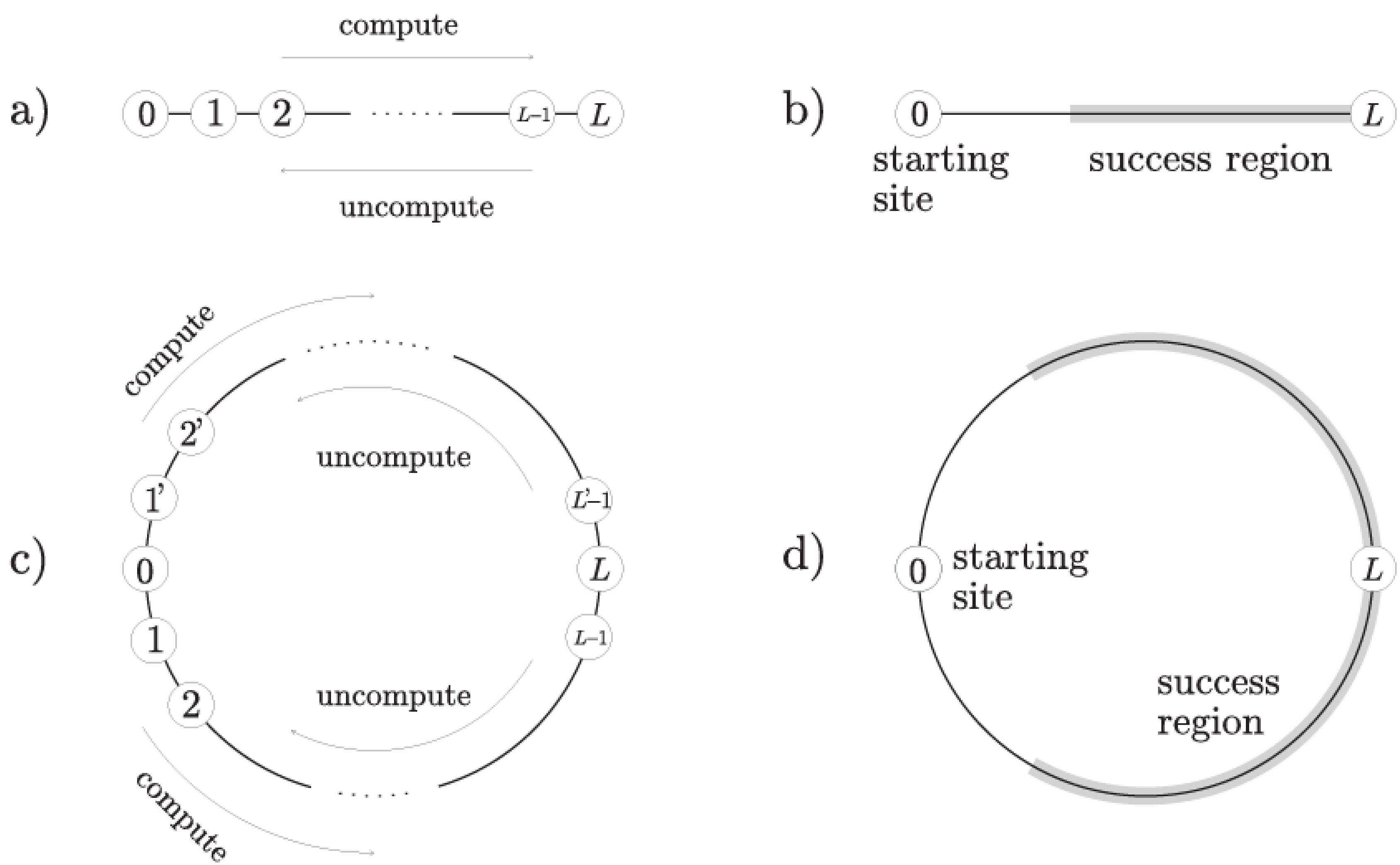
a possibility: wrap around a circle

$$\begin{aligned} &|\varphi_0\rangle \otimes |0\rangle_c \\ U_1 &|\varphi_0\rangle \otimes |1\rangle_c \\ U_2 U_1 &|\varphi_0\rangle \otimes |2\rangle_c \\ U_3 U_2 U_1 &|\varphi_0\rangle \otimes |3\rangle_c \\ U_4 U_3 U_2 U_1 &|\varphi_0\rangle \otimes |4\rangle_c \end{aligned}$$

$$\boldsymbol{H_F} = - \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

  – the eigenvectors: walk on a line, plain waves

2a) **Boosting the success probability**

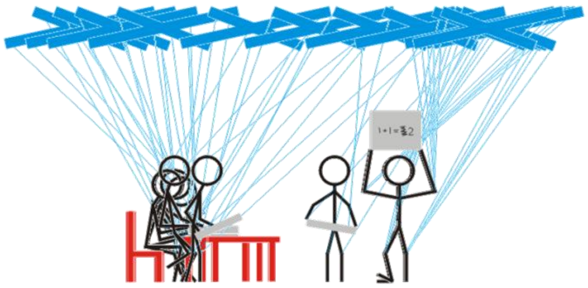


3) The Local Hamiltonian problem

- **The history state**
  - a state encoding the progress of a quantum computation

$$|\psi_{hist}\rangle = \sum_{t=0}^{L} (U_t \ldots U_1) |\varphi_0\rangle \otimes |t\rangle_c$$

work register after t gates



  - encodes also the result of $U|\varphi_0\rangle$

- **A ground state**
  - a Hamiltonian with energy penalties for
    - non-history states (bad computation)
    - states with computations yielding `no' $\quad |no\rangle \otimes |L\rangle_c$
  - if a circuit can output `yes', a `good' history state exists
  - the ground state of H then has low energy

3) **The Local Hamiltonian problem**

- **The history state:** a ground state
  - a state encoding the progress of a quantum computation

$$|\psi_{hist}\rangle = \sum_{t=0}^{L} (U_t \ldots U_1) |\varphi_0\rangle \otimes |t\rangle_c$$

$$|\varphi\rangle \quad \otimes \quad |t-1\rangle_c$$

$$U_t |\varphi\rangle \quad \otimes \quad |t\rangle_c$$

- **Kitaev's (k-)Local Hamiltonian**

$$H = \sum_{t=1}^{L} H_t + H_{in} + H_{out}$$

computation (history)

$$\begin{aligned} H_t &= \frac{1}{2}\left(|t\rangle\langle t|_c + |t-1\rangle\langle t-1|_c\right) \\ &- \frac{1}{2}\left(U_t \otimes |t\rangle\langle t-1| + U_t^\dagger \otimes |t-1\rangle\langle t|\right) \end{aligned} \qquad = \frac{1}{2}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

3) **The Local Hamiltonian problem**

- The propagation Hamiltonian

$$H_t = \frac{1}{2}\left(|t\rangle\langle t|_c + |t-1\rangle\langle t-1|_c\right) - \frac{1}{2}\left(U_t \otimes |t\rangle\langle t-1| + U_t^\dagger \otimes |t-1\rangle\langle t|\right)$$

$$= \frac{1}{2}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

in some basis

- A good basis: the "line" of states

$$\begin{array}{r} |\varphi_0\rangle \otimes |0\rangle_c \\ U_1|\varphi_0\rangle \otimes |1\rangle_c \\ U_2U_1|\varphi_0\rangle \otimes |2\rangle_c \\ U_3U_2U_1|\varphi_0\rangle \otimes |3\rangle_c \\ U_4U_3U_2U_1|\varphi_0\rangle \otimes |4\rangle_c \end{array}$$

$$\sum_{t=1}^{L} H_t = \frac{1}{2}\begin{bmatrix} 1 & -1 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & -1 & 2 & -1 \\ 0 & 0 & 0 & -1 & 1 \end{bmatrix}$$

  – a positive operator (a sum of projectors)
  – the only zero-energy state: the uniform superposition

3) **The Local Hamiltonian problem**

- **The history state**: a ground state
  - a state encoding the progress of a quantum computation

$$|\psi_{hist}\rangle = \sum_{t=0}^{L} (U_t \dots U_1) |\varphi_0\rangle \otimes |t\rangle_c$$

$$|\varphi_0\rangle \otimes |0\rangle_c$$

check that the ancilla qubits are not 1

- **Kitaev's (k-)Local Hamiltonian**

$$H = \sum_{t=1}^{L} H_t + H_{in} + H_{out}$$

initialization

3) The Local Hamiltonian problem

- **The history state**: a ground state
  - a state encoding the progress of a quantum computation

$$|\psi_{hist}\rangle = \sum_{t=0}^{L} (U_t \ldots U_1) |\varphi_0\rangle \otimes |t\rangle_c$$

$$|\varphi_L\rangle \otimes |L\rangle_c$$

check that the final answer is not 0

- **Kitaev's (k-)Local Hamiltonian**

$$H = \sum_{t=1}^{L} H_t + H_{in} + H_{out}$$

final answer

3) The Local Hamiltonian problem

- **The history state**: a ground state
  - a state encoding the progress of a quantum computation

$$|\psi_{hist}\rangle = \sum_{t=0}^{L} (U_t \dots U_1) |\varphi_0\rangle \otimes |t\rangle_c$$

- **Kitaev's (k-)Local Hamiltonian**

$$H = \sum_{t=1}^{L} H_t + H_{in} + H_{out}$$

  - encoding the clock: in binary addressing, transitions: log-L local
  - gives a **(log-L)**-local Hamiltonian problem

$$\begin{aligned} |t=0\rangle_c &= |0000\rangle \\ |t=1\rangle_c &= |0001\rangle \\ |t=2\rangle_c &= |0010\rangle \\ |t=3\rangle_c &= |0011\rangle \\ |t=4\rangle_c &= |0100\rangle \end{aligned}$$

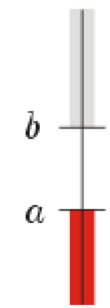
3) The Local Hamiltonian problem

- **The history state**: a ground state
  - a state encoding the progress of a quantum computation

$$|\psi_{hist}\rangle = \frac{1}{\sqrt{L+1}} \sum_{t=0}^{L} (U_t \dots U_1) |\varphi_0\rangle \otimes |t\rangle_c$$

- **Kitaev's (k-)Local Hamiltonian**

$$H = \sum_{t=1}^{L} H_t + H_{in} + H_{out}$$



  - **YES** case: there exists a state $|\varphi_0\rangle$ accepted by U with probability 1-ε
  - its history state has energy

$$\langle\psi_{hist}| H |\psi_{hist}\rangle \leq 0 + 0 + \frac{\epsilon}{L+1}$$

3) The Local Hamiltonian problem

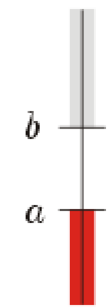
– **NO** case: any state is accepted by U with probability at most ε

– any state has energy
(... details in Part II ...)

$$\langle\eta| H |\eta\rangle \geq \frac{c\,(1-\sqrt{\epsilon})}{L^3}$$

- **Kitaev's (k-)Local Hamiltonian**

$$H = \sum_{t=1}^{L} H_t + H_{in} + H_{out}$$



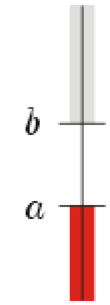
– **YES** case: there exists a state $|\varphi_0\rangle$ accepted by U with probability 1-ε

– its history state has energy

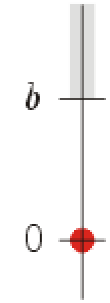
$$\langle\psi_{hist}| H |\psi_{hist}\rangle \leq 0 + 0 + \frac{\epsilon}{L+1}$$

3) Local Hamiltonian and Quantum k-SAT

- **Local Hamiltonian** [Kitaev]
  - an analogue of classical MAX-k-SAT
  - is the ground state energy of the whole H less than a or more than b?



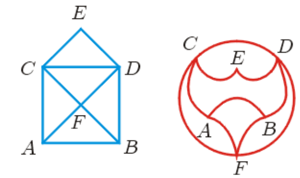
- **Quantum k-SAT** [Bravyi]
  - an analogue of classical k-SAT
  - Hamiltonian: a sum of projectors. Can they all be satisfied? $H = \sum_i P_i$



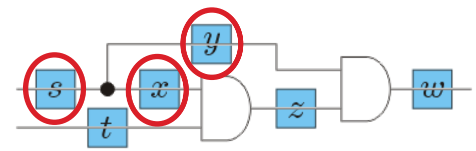
- How to prove they are hard?
  - encode any q. computation U into the ground state of some H
  - knowing the ground state energy of H means knowing whether U can ever output `yes'

3) **Conclusions [Part I]**

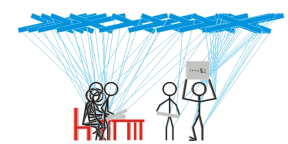
- **hard yes/no problems (NP, QMA)**
  - easy to verify a (classical, quantum) “proof”



- **3-SAT is NP-complete**
  - a successful computation
    = a satisfiable instance of 3-SAT

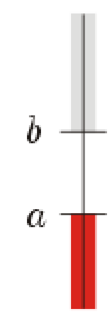


- **encoding a quantum computation into a state**
  - Feynman’s Hamiltonian, the history state



- **making a Hamiltonian problem**
  - low ground state energy
    = existence of a “proof”

$$H = \sum_{t=1}^{L} H_t + H_{in} + H_{out}$$

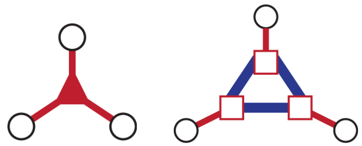


  - (log-L) Local Hamiltonian [Kitaev]
    - sound and complete: QMA-complete

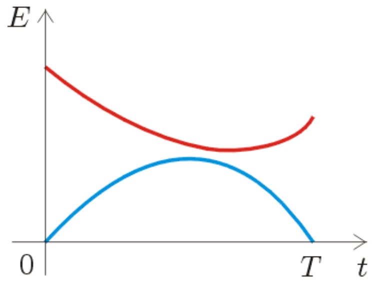
Part II [preview]

- from log-L local to **5-local Hamiltonian**
  - domain wall clock + more details…

- going down to **2-local**:
  - perturbation gadgets, 2D grid
  - Quantum 2-SAT on a line (1D)
  - Quantum (5,3)-SAT



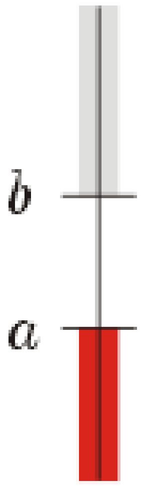
- making quantum computers from local Hamiltonians
  - **Adiabatic Quantum Computation**
  - constructing clocks with nonlinear time progression (triangle, switches…)



time for a coffee break...

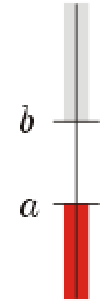
1) **Local Hamiltonian is in QMA**

- what's the yes/no question?
  - is the g.s. energy low?
- what's the witness?
  - the ground state
- how do you verify it?
  - randomly pick one of the conditions... (for Q-k-SAT)



3a) **Local Hamiltonian is in QMA**

- **How do you show it? Here's a circuit...**
  - the verification circuit accepts:
    - YES case: a good witness with probability $p \geq 1 - \frac{a}{L}$
    - NO case: any state with $p \leq 1 - \frac{b}{L}$



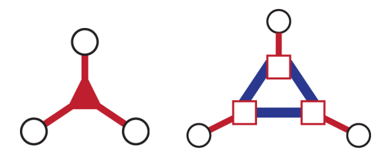
  - for each term, write $H_j = \sum_s \lambda_s \left|\psi_s\right\rangle \left\langle\psi_s\right|$

    make the circuit $W_j \left|\psi_s\right\rangle \left|0\right\rangle_B = \left|\psi_s\right\rangle \left(\sqrt{\lambda_s} \left|0\right\rangle_B + \sqrt{1-\lambda_s} \left|1\right\rangle_B\right)$
  - use a controlled version of W: choose j according to another register
  - run the circuit on $\frac{1}{\sqrt{L}} \sum_{j=1}^{L} \left|j\right\rangle_A \left|\eta\right\rangle \left|0\right\rangle_B$
  - the probability to measure 1 on ancilla B is $p = 1 - \frac{\left\langle\eta\right| H \left|\eta\right\rangle}{L}$

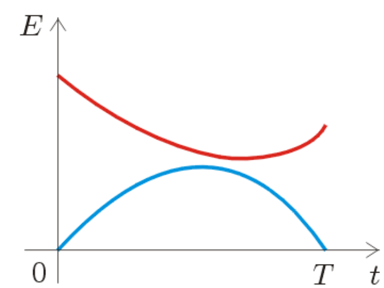
Part II

- from log-L local to **5-local Hamiltonian**
  - domain wall clock + more details...

- going down to **2-local**:
  - perturbation gadgets, 2D grid
  - Quantum 2-SAT on a line (1D)
  - Quantum (5,3)-SAT



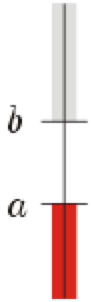
- making quantum computers from local Hamiltonians
  - **Adiabatic Quantum Computation**
  - constructing clocks with nonlinear time progression (triangle, switches...)



1) The 5-Local Hamiltonian

- **Kitaev's (k-)Local Hamiltonian**

$$H = \sum_{t=1}^{L} H_t + H_{in} + H_{out}$$



  - is the ground state energy of H less than a or more than b?
  - (log-L)-local Hamiltonian: QMA-complete

- **A better clock encoding: better locality**
  - a domain-wall clock

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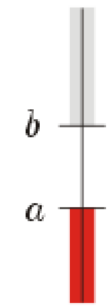

  - is the ground state energy of H less than a or more than b?
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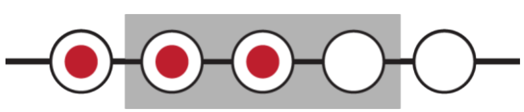
- **A better clock encoding: better locality**
  - a domain-wall clock
  - clock addressing: 3-local

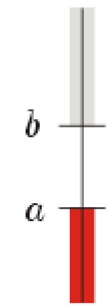


$$\begin{aligned} H_t &= \frac{1}{2}\left(|t\rangle\langle t|_c + |t-1\rangle\langle t-1|_c\right) \\ &- \frac{1}{2}\left(U_t \otimes |t\rangle\langle t-1| + U_t^\dagger \otimes |t-1\rangle\langle t|\right) \end{aligned}$$

1) The 5-Local Hamiltonian

- **Kitaev's (k-)Local Hamiltonian**

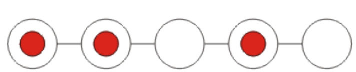
$$H = \sum_{t=1}^{L} H_t + H_{in} + H_{out} + H_{clock}$$



  - is the ground state energy of H less than a or more than b?
  - (log-L)-local Hamiltonian: QMA-complete

- **A better clock encoding: better locality**
  - a domain-wall clock
  - clock addressing: 3-local
  - bad clock states? penalize them!

$$H_{clock}^{(t)} = |01\rangle\langle 01|_{t-1,t}$$



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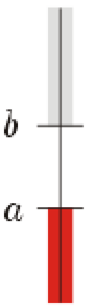


  - is the ground state energy of H less than a or more than b?
  - (log-L)-local Hamiltonian: QMA-complete

- **A better clock encoding: better locality**
  - a domain-wall clock

  - clock addressing: 3-local
  - clock checking: 2-local

$$H_{clock}^{(t)} = |01\rangle \langle 01|_{t-1,t}$$

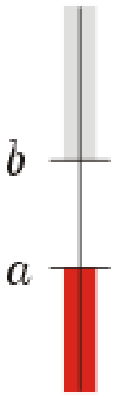
  - 3 (address) + 2 (gate) = **5-local Hamiltonian** is QMA-complete

1a) **Local Hamiltonian: Soundness**

- the lowest eigenvalue of H is high in the **NO** case
  - no solution exists ... no “happy” computation (accept < ε)
  - the Kitaev Hamiltonian

$$H = \sum_{t=1}^{L} H_t + H_{in} + H_{out} + H_{clock}$$

  - any state now
    - has a bad clock
    - is not a good history state (doesn’t compute correctly)
    - is unlikely to be accepted at the end
  - its energy is “high” (gives a lower bound on b)



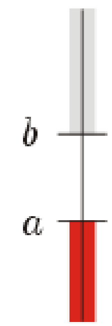
1a) **Local Hamiltonian: Soundness**

- a geometric lemma [Kitaev, Shen, Vialyi] $A, B \geq 0$
  - the sum of two nonnegative operators with no common **null space** is lower bounded:

$$A + B \geq \delta \cdot 2\sin^2 \frac{\vartheta}{2}$$

    - the second eigenvalues of A, B are greater than δ
    - depends on the angle ϑ between the null subspaces of A and B
    - A - the clock and initialization: $\delta \geq 1$

      B - the propagation: $\delta \geq \frac{c}{L^2}$



  - the ground state energy is high in the **NO** case

2) **Going down to 3-local**

- Domain wall clock

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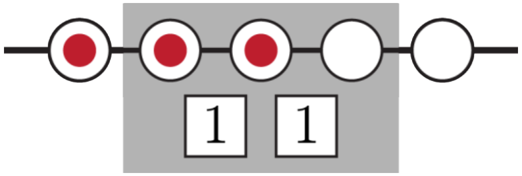
- Domain wall clock

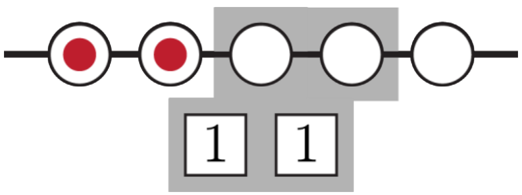


transitions: 3-local
2-qubit gates: 5-local

  - used by Kitaev (5-local Hamiltonian)
  - easy to check initialization, output, single active site

2) **Going down to 3-local**

- Domain wall clock



transitions: 3-local
2-qubit gates: 5-local

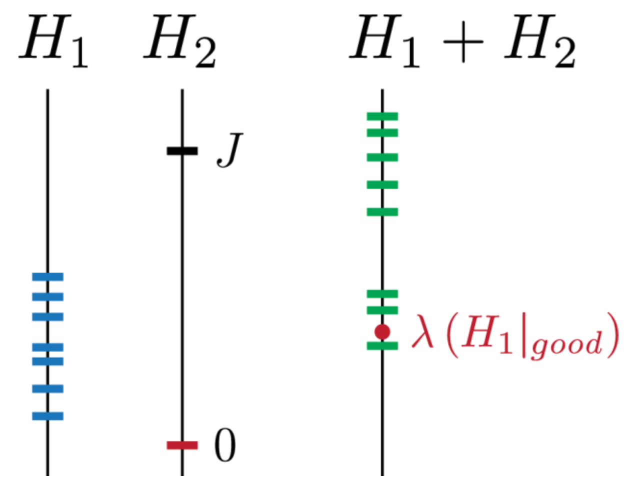
  - used by Kitaev (5-local Hamiltonian is QMA_1-complete)
  - easy to check initialization, output, single active site

- 3-local Hamiltonian [Kempe & Regev]
  - suppressing bad transitions: projection lemma

- 2-local Hamiltonian [Kempe, Kitaev, Regev, Oliveira & Terhal]
  - effective 3-local interactions: gadgets, even in 2D

2b) **Local Hamiltonian: Killing bad clocks**

- the projection lemma [Kempe, Kitaev, Regev]
  - the sum of two operators
    - a "good" subspace *(where H_2 is 0)*
    - a "bad" subspace *(where H_2 is big = J)*
  - the lowest eigenvector of the sum lies close to the "good" subspace



Lemma 1 *Let $H = H_1 + H_2$ be the sum of two Hamiltonians operating on some Hilbert space $\mathcal{H} = \mathcal{S} + \mathcal{S}^\perp$. The Hamiltonian H_2 is such that $\mathcal{S}$ is a zero eigenspace and the eigenvectors in $\mathcal{S}^\perp$ have eigenvalue at least $J > 2\|H_1\|$. Then,*

$$\lambda(H_1|_\mathcal{S}) - \frac{\|H_1\|^2}{J - 2\|H_1\|} \leq \lambda(H) \leq \lambda(H_1|_\mathcal{S}).$$

2b) **Local Hamiltonian: Killing bad clocks**

- the projection lemma [Kempe, Kitaev, Regev]
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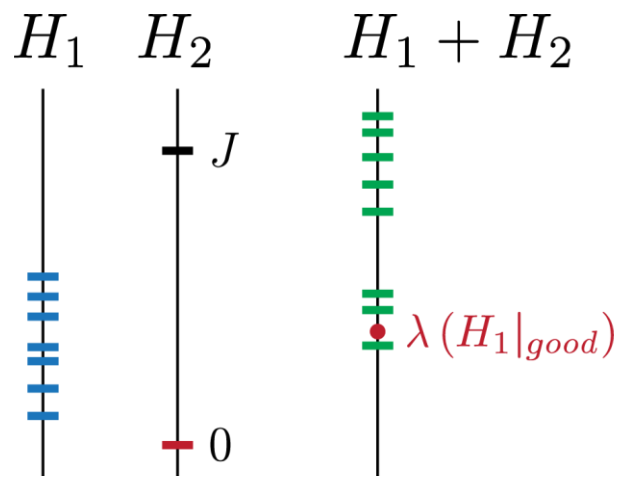

  - the lowest eigenvector of the sum lies close to the "good" subspace
  - makes an **effective** Hamiltonian

2b) **Local Hamiltonian: Killing bad clocks**

- the projection lemma [Kempe, Kitaev, Regev]
  - the sum of two operators
    - a "good" subspace
      (where H_2 is 0)
    - a "bad" subspace
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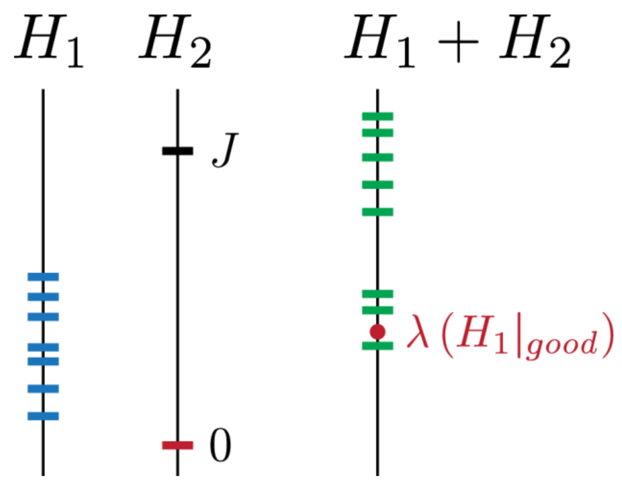
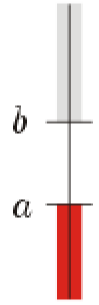


  - the lowest eigenvector of the sum lies close to the "good" subspace
  - application 1: a bigger separation *(b-a)* for Kitaev's *(log-L)*-local Hamiltonian (at a cost...)



$$H = H_{out} + J_{in} H_{in} + J_{prop} H_{prop}$$

2b) **Local Hamiltonian: Killing bad clocks**

- the projection lemma [Kempe, Kitaev, Regev]
  - the sum of two operators
    - a "good" subspace
      (where H_2 is 0)
    - a "bad" subspace
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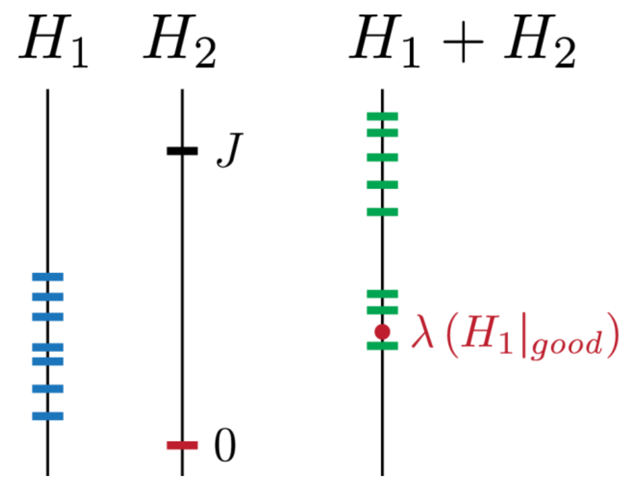
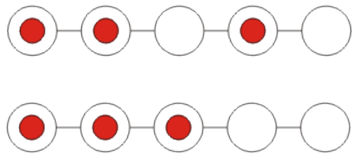


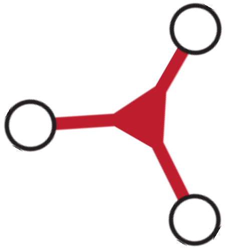
  - the lowest eigenvector of the sum lies close to the "good" subspace
  - application 2: from 5-local to 3-local Hamiltonian
    - "bad" clock states cannot have low eigenvalues
    - "good" clock states stay OK



2) **Going down to 2-local**

- effective 3-local interactions from 2-local gadgets [KKR]
  - start: a 3-local Hamiltonian

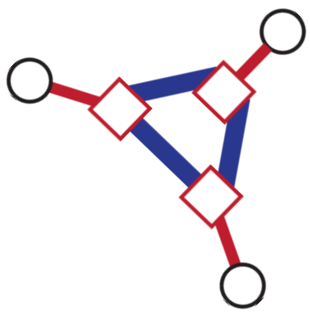
    $$H_0 = Y - B_1 B_2 B_3$$



  - add 3 ancillae, "force" them to 000/111 with

    $$H = -\frac{1}{4\delta^3}\left(\sigma_1^z\sigma_2^z + \sigma_2^z\sigma_3^z + \sigma_3^z\sigma_1^z - 3\mathbb{I}\right)$$

    interact with the outside qubits



$$W = -\frac{1}{\delta^2}\left(B_1 \otimes \sigma_1^x + B_2 \otimes \sigma_2^x + B_3 \otimes \sigma_3^x\right)$$

$$V = Y + \frac{1}{\delta}\left(B_1^2 + B_2^2 + B_3^2\right)$$

  - the resulting Hamiltonian $\tilde{H} = H + V + W$ is effectively close to H_0

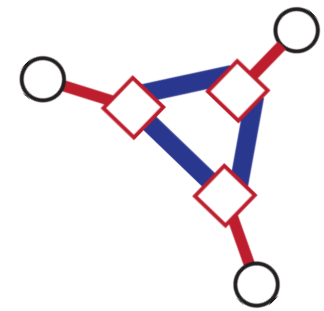
2) **Going down to 2-local**

- 3-local effective interactions from 2-local gadgets [KKR]

$$H = -\frac{1}{4\delta^3}\left(\sigma_1^z\sigma_2^z + \sigma_2^z\sigma_3^z + \sigma_3^z\sigma_1^z - 3\mathbb{I}\right)$$

$$W = -\frac{1}{\delta^2}\left(B_1 \otimes \sigma_1^x + B_2 \otimes \sigma_2^x + B_3 \otimes \sigma_3^x\right)$$

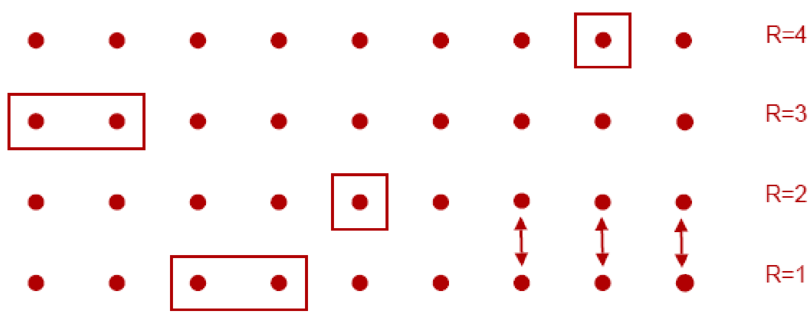
$$V = Y + \frac{1}{\delta}\left(B_1^2 + B_2^2 + B_3^2\right)$$



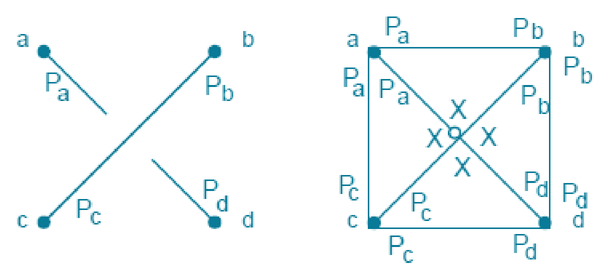
- the Hamiltonian $\tilde{H} = H + V + W$ is effectively what we want (generated by higher order terms in perturbation theory)
- the (energy) cost: huge penalty terms (high norms of H, V, W)
- huge? polynomial in L (the size of the circuit)
- conversion to adiabatic QC? Far from practical...

2) **Going down to 2-local**

- 2-local interactions on a 2D grid [Oliveira, Terhal]
  - start with a 5-local Hamiltonian, make it spatially sparse (push qubits up a grid after gates are done)



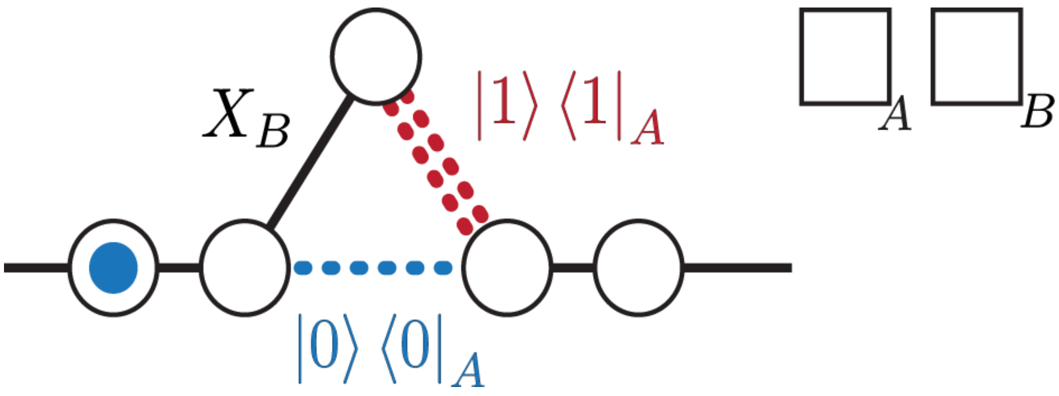
  - reduce it to 3-local (punish bad clocks)
  - go to 2-local, planar, square lattice using many mediator qubits



  - again, all this at a high (polynomial) energy cost...

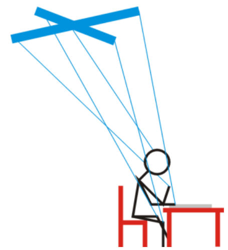
2a) **Going down to 2-local: Quantum 2-SAT**

- Quantum 2-SAT with qudits (5,3) [Eldar, Regev]

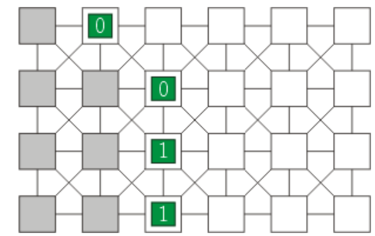


2a) **Local Hamiltonians in 1D**

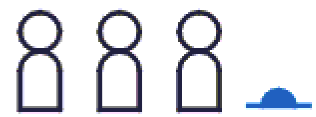
- two registers (clock/work)



- geometric clock

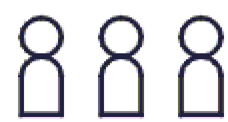


  – geometric clock, Q-2-SAT in 1D [Aharonov et al.]



    - Quantum 2-SAT on a line of qudits (d=11) is QMA_1 complete

  – diffusion clock [Cirac et al.]



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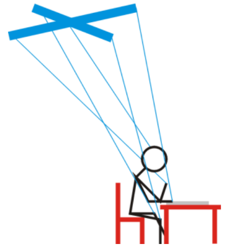
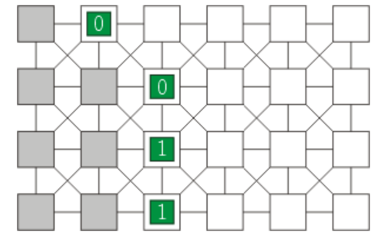


- geometric clock



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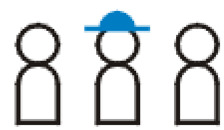
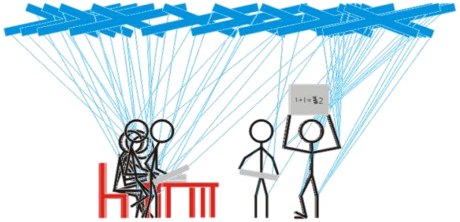


    - Quantum 2-SAT on a line of qudits (d=11) is QMA_1 complete

  - a translationally invariant system, HQCA [Nagaj & Wocjan]

3) **BQP universality**

- can we use an LH to build a quantum computer?
  - the second question about local Hamiltonians
    (the first: are their properties hard to determine?)
  - the answer? YES! [AKvDLLR]
  - using adiabatic quantum computing [Farhi et al.]
- two ways to achieve BQP universality
  - models that adiabatically
    prepare the history state
  - walk-based *(not necessarily adiabatic)*
    models preparing the final output



3) **Adiabatic Quantum Computing**

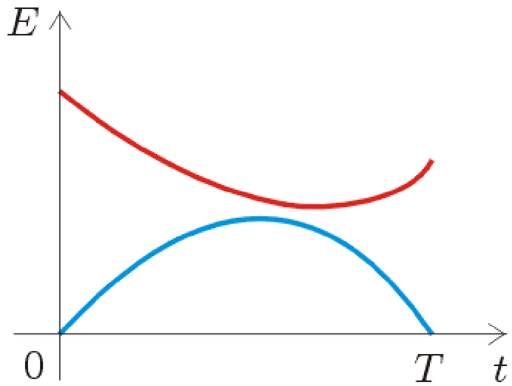
- Ground states and optimization problems
  - a cost function $h(z)$ of an optimization problem

$$H_P |z\rangle = h(z) |z\rangle$$

- A Hamiltonian Algorithm [Farhi et al.]
  - use a time-dependent, slowly changing Hamiltonian

$$H(t) = \left(1 - \frac{t}{T}\right) H_B + \frac{t}{T} H_P$$

- Adiabatic Theorem
  - start in the ground state, end up in the ground state
  - how slow is "slow"?



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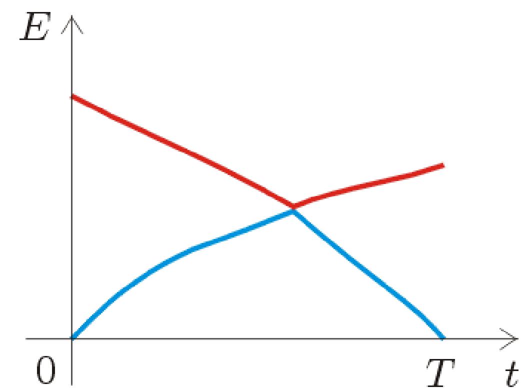


3) **BQP universality using AQC**

$$H(t) = \left(1 - \frac{t}{T}\right) H_B + \frac{t}{T} H_P$$

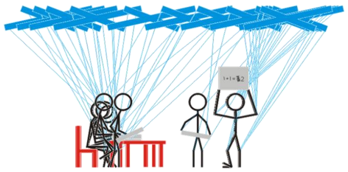
- Start: fix the initialization
  - ground state: $|\varphi_0\rangle\, |0\rangle_c$

$$\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

- End: Kitaev's propagation Hamiltonian
  - ground state: the history state

$$\frac{1}{2}\begin{bmatrix} 1 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 1 \end{bmatrix}$$

- end up close to the history state
  - the work: show that the gap is polynomial
  - AQC is universal for BQP [AvDKLLR]



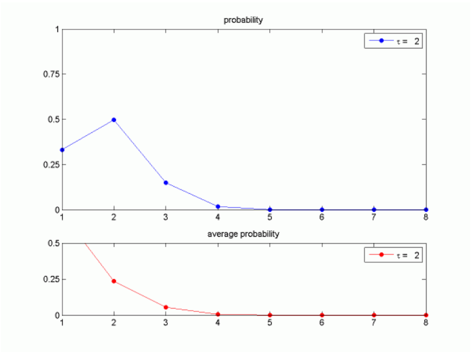
3) **BQP universality using a quantum walk**

- Unique transitions
  - stay within a computational subspace

$$|\psi_t\rangle \quad = \quad (U_t \ldots U_2 U_1)\,|\varphi_0\rangle \otimes |t\rangle_c$$

- The Hamiltonian
  - start by fixing the initial state
  - quickly switch on the Feynman Hamiltonian and let the system run
  - the dynamics: **a quantum walk on a line**

$$\begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$



3) **Constructing Clocks: Linear Time**

- Domain wall clock

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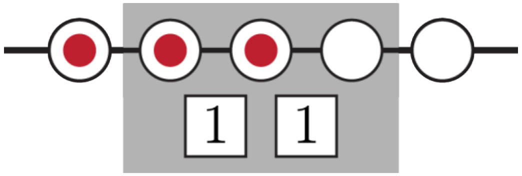


transitions: 3-local
2-qubit gates: 5-local

  - used by Kitaev (5-local Hamiltonian)
  - easy to check initialization, output, single active site

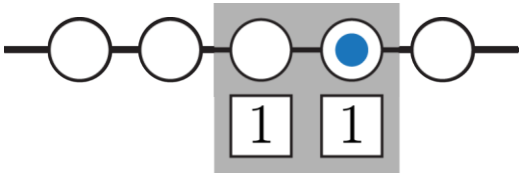
3) **Constructing Clocks: Linear Time**

- Pulse clock

  - Feynman's pointer particle idea

3) Constructing Clocks: Linear Time

- Pulse clock

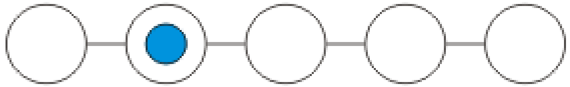


transitions: 2-local
2-qubit gates: 4-local

  - Feynman's pointer particle idea
  - needs initialization
    - the dead state problem: bad for Quantum k-SAT
  - good for BQP universality
    - 4-local (vs. 5-local)
    - no bad transitions
    - no need to go slowly for adiabatic

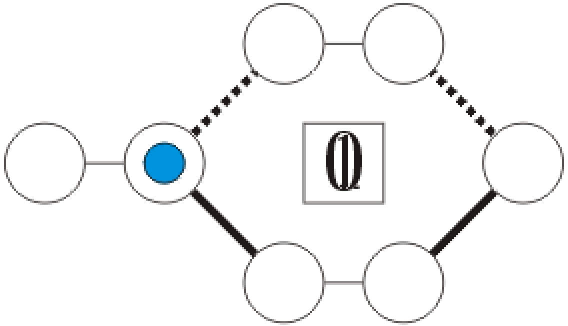
3) **Constructing Clocks: Railroad Switch**

- One train, two tracks



3) **Railroad Switch**

- One train, two tracks



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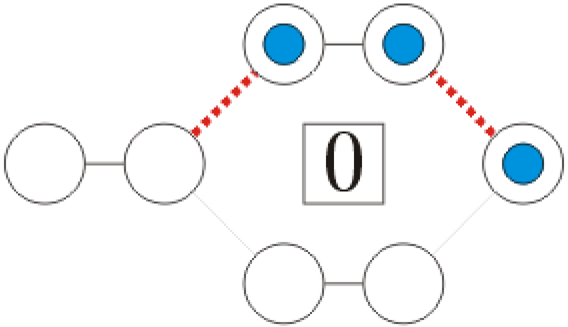

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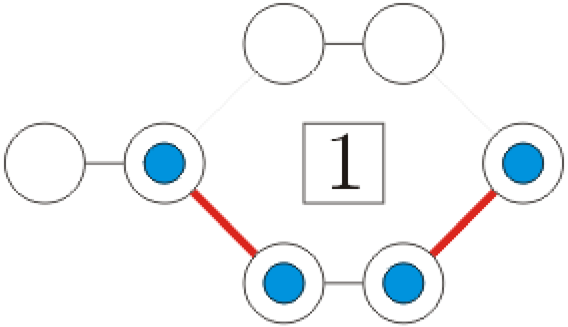

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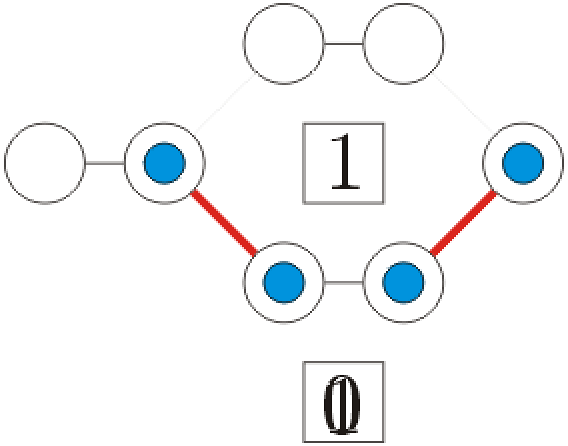

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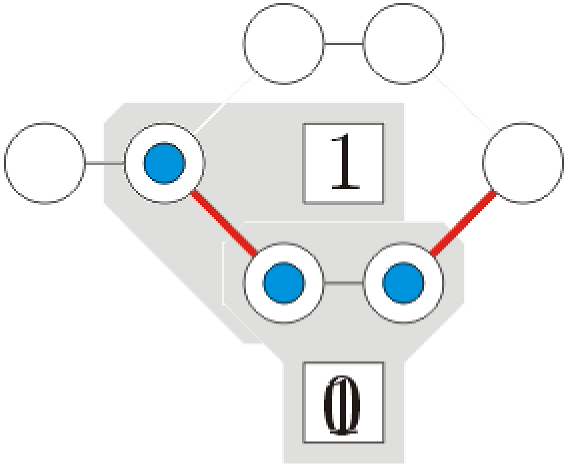
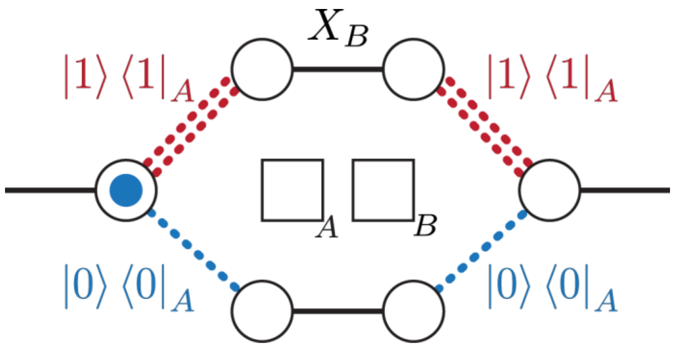


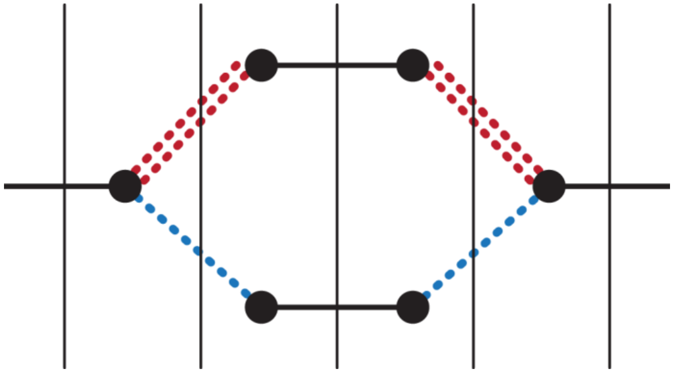
transitions: 3
gates: 3

3) **Railroad Switch**

- One train, two tracks



- The computational subspace: a line again!



$$|1\rangle = (\alpha|0\rangle_A|\varphi\rangle_B + \beta|1\rangle_A|\varphi\rangle_B) \otimes |1\rangle_c$$

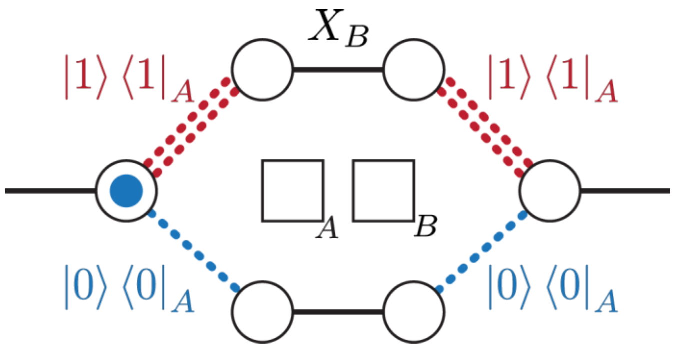
$$|2\rangle = \alpha|0\rangle_A|\varphi\rangle_B \otimes |2u\rangle_c + \beta|1\rangle_A|\varphi\rangle_B \otimes |2d\rangle_c$$

$$|3\rangle = \alpha|0\rangle_A X_2|\varphi\rangle_B \otimes |3u\rangle_c + \beta|1\rangle_A|\varphi\rangle_B \otimes |3d\rangle_c$$

$$|4\rangle = (\alpha|0\rangle_A|\varphi\rangle_B + \beta|1\rangle_A X_2|\varphi\rangle_B) \otimes |4\rangle_c$$

3) **Universality of Quantum 3-SAT**

- Using a railroad switch clock
  - fast, universal quantum computation with a Q-3-SAT Hamiltonian
  - made from 3-local projectors



[Nagaj, 2008]

3) **Universality of Quantum 3-SAT**

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  - fast, universal quantum computation with a Q-3-SAT Hamiltonian
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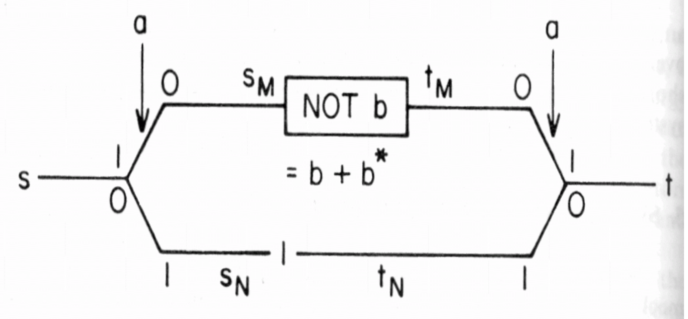
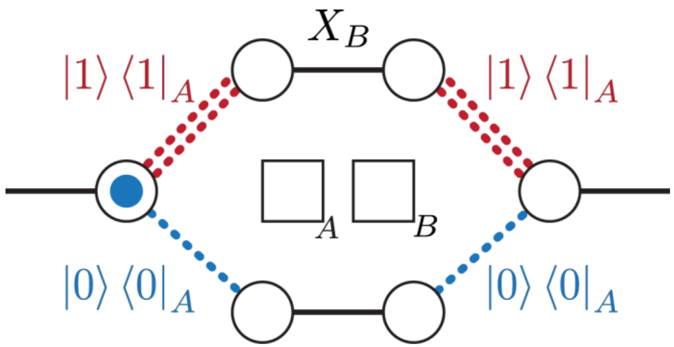


Fig. 8. CONTROLLED NOT by switches.

[Feynman, 1985]

3) **Universality of Quantum 3-SAT**

- Using a railroad switch clock
  - fast, universal quantum computation with a Q-3-SAT Hamiltonian

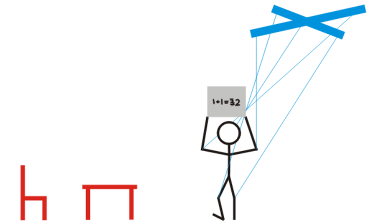


  - made from 3-local projectors
  - resources: $\tau \cdot \|H\| = L \log^2 L \times L = L^2 \log^2 L$
  - the computational subspace
    - protected by a gap O(L⁻¹)
    - not against everything (loss of a pointer)

$$|\varphi_0\rangle \otimes |0\rangle_c$$
$$U_1 |\varphi_0\rangle \otimes |1\rangle_c$$
$$U_2U_1 |\varphi_0\rangle \otimes |2\rangle_c$$
$$U_3U_2U_1 |\varphi_0\rangle \otimes |3\rangle_c$$

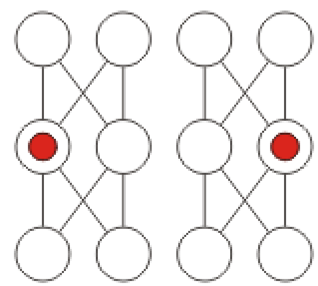
3) **Efficient Simulation of Quantum Circuits**

- Use a Hamiltonian Computer

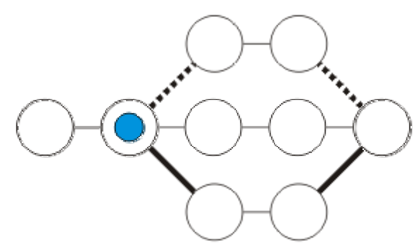


  - [AvDKLLR]: AQC is universal
    3-local, L^{17}

  - [Mizel,Lidar]: AQC is universal
    4-loc,al L^{4}

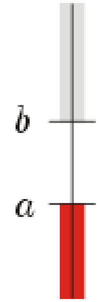


  - use a better one [Nagaj,Mozes]
    3-local, L^{7}
  - go fast! [Lloyd]
  - railroad switch: 3-local, L^{2} [Nagaj]
    - certified fresh: 2-local, same scaling...

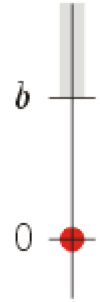


4) State of the art

- **MAX-k-SAT**
  - NP-complete for k≥2
    - MAX-2-sat



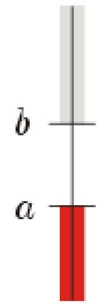
- **k-SAT**
  - easy for k=2
  - NP-complete for k≥3
    - 3-SAT



  - with dits
    - (3,2)-SAT is NP-complete
    - simple in 1D for all dits

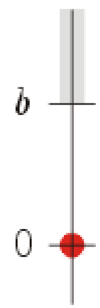
4) State of the art

- **MAX-k-SAT**
  - NP-complete for k≥2
    - MAX-2-sat



- **k-local Hamiltonian**
  - QMA-complete for k≥2
    - 2-local Ham, even in 2D

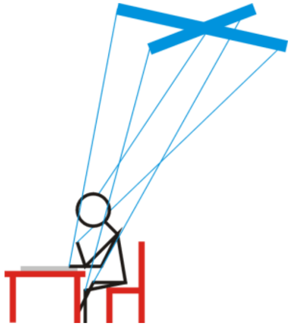
- **k-SAT**
  - easy for k=2
  - NP-complete for k≥3
    - 3-SAT
  - with dits
    - (3,2)-SAT is NP-complete
    - simple in 1D for all dits



- **Quantum-k-SAT**
  - easy for k=2
  - QMA_1-complete for k≥4
    - k=4, using 3-local projectors
  - universal: Quantum-3-SAT
  - with qudits
    - QMA_1-complete: Q-(5,3)-SAT
    - universal: Q-(3,2)-SAT
    - QMA_1-c.: Q-(11,11)-SAT in 1D

4) **Conclusions [Part I + II]**

- Are local Hamiltonians good for something?
  - universality (adiabatic/nonadiabatic) results
    - domain wall/pulse clock constructions...
- Tough questions about physical systems?
  - is the ground state energy low? *(it's hard to cool it)*
  - QMA complete problems *(LH, Q-k-SAT)*
    - perturbation gadgets, 1D problems with qudits...
- Simple-looking problems that are still crazy hard?
    - Quantum 3-SAT?
    - Lower-d Q-2-SAT in 1D? [Movassagh et al.]
    - translational invariance & tiling [Gottesman, Irani]
  - Quantum PCP?
    - large b-a: still hard?



enough local Hamiltonians, enjoy local ...

www.tripadvisor.com