Fusion rules and boundary conditions in logarithmic CFT

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joint work with

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Outline

• From boundary to bulk in non-logarithmic rational CFT

• The $W_{2,3}$ model with $c=0$

• A boundary theory for the $W_{2,3}$ model

• (A few words on the bulk theory)
Boundary CFT - the Cardy case (non-log., rational)

e.g. Virasoro minimal models $V(p,q)$

irreducible representations $h_{r,s}$
elementary boundary conditions

\[ \{ \text{Kac label } (r,s) \} \]

Cylinder partition function

\[
A_{a,b}(q) = \sum_c N_{ab}^c \chi_c(q) \quad q = e^{-\pi L/R}
\]

\[
U_a \otimes U_b \cong \bigoplus_c N_{ab}^c U_c
\]

Rewrite: $A_{U,V}(q) = \chi_V \otimes U^*(q)$
Boundary is simpler than bulk (non-log., rational)

Bulk partition function:

$$Z(q) = \sum |\chi_i(q)|^2$$

Cylinder partition function for $a=b=(1,1)$:

$$A_{a,a}(q) = \chi(1,1)(q)$$

The space of states on the $(1,1)$-boundary is the irreducible Virasoro vacuum representation ($h=0$)
From boundary to bulk

1) Disc correlator of one bulk and one boundary field non-degenerate:

\[ \phi \cdot \psi = 0 \quad \text{for all } \psi \quad \text{then } \phi = 0 \]

Want this, because:

\[ \phi_1 \cdot \phi_2 = \sum_{\psi} \phi_1 \cdot \psi \cdot \psi \cdot \phi_2 \]
...from boundary to bulk

2) Disc correlator symmetric:
...from boundary to bulk

3) Take biggest $\mathcal{H}_{\text{bulk}}$ satisfying 1) and 2)

(non-log., rational)

e.g. boundary condition labelled by vacuum rep. $(1,1)$

- space of boundary fields is the VOA itself
  
  $$\mathcal{H}_{(1,1)\rightarrow(1,1)} = \mathcal{V}$$

- ansatz for space of bulk fields
  
  $$\mathcal{H}_{\text{bulk}} = \bigoplus_{i,j} Z_{ij} U_i \otimes_{\mathbb{C}} \bar{U}_j^*$$

- constrain $Z_{ij} \ldots$
...from boundary to bulk

\[ \mathcal{H}_{\text{bulk}} = \bigoplus_{i,j} Z_{ij} U_i \otimes \mathbb{C} \bar{U}_j^* \]

\[ \psi \in \mathcal{H}_{(1,1) \to (1,1)} = \mathcal{V} \]

\[ \langle \phi \psi \rangle = 0 \]

(3-point conformal block on the sphere vanishes for insertions of \( U_i, U_j^* \) and \( \mathcal{V} \) if \( i \neq j \).)
...from boundary to bulk

\[ \mathcal{H}_{\text{bulk}} = \bigoplus_i Z_{ii} U_i \otimes \mathbb{C} \overline{U}_i^* \]

- \( Z_{ii} \leq 1 \): Suppose

\[ \mathcal{H}_{\text{bulk}} = \cdots \bigoplus (U_i \otimes \mathbb{C} \overline{U}_i^*)_1 \bigoplus (U_i \otimes \mathbb{C} \overline{U}_i^*)_2 \bigoplus \cdots \]

Take \( \phi_1 \in (U_i \otimes \mathbb{C} \overline{U}_i^*)_1 \), \( \phi_2 \in (U_i \otimes \mathbb{C} \overline{U}_i^*)_2 \), then

\[ \langle \phi_1 \psi \rangle = \lambda_1 \cdot b(\phi_1, \psi) \quad \langle \phi_2 \psi \rangle = \lambda_2 \cdot b(\phi_2, \psi) \]

for the same \( b : (U_i \otimes \mathbb{C} \overline{U}_i^*) \times \mathcal{V} \rightarrow \mathbb{C} \).

Hence, \( \lambda_2 \phi_1 - \lambda_1 \phi_2 \) vanishes in all disc correlators.

(non-log., rational)
Results from rational CFT

• **Given a boundary theory** (boundary labels, spaces of boundary fields, boundary OPEs) there **exists a unique bulk theory** that fits to this boundary theory.

• The bulk theory is determined by 1) - 3) above.

• **Every** bulk theory with the same holomorphic and anti-holomorphic rational chiral algebra can be obtained in this way.

Fjelstad, Fuchs, Schweigert, IR ‘06
Kong, IR ‘08
Rational logarithmic CFT

Study the $W_{2,3}$ triplet model with central charge $c=0$.

(because the $W_{1,p}$ models are “too simple”)

The $W_{23}$ model

- Virasoro Verma module for $h=0$ and $c=0$:
  two independent null vectors
  \[ N_1 = L_{-1} \Omega \quad N_2 = (L_{-2} - \frac{3}{2} L_{-1} L_{-1}) \Omega \]

- Divide by $N_1$ and $N_2$: get
  \[ \mathcal{V}(0) = C \cdot \Omega \]

- Divide by $N_1$ but not by $N_2$: get $\mathcal{V}$ with character
  \[ \chi_{\mathcal{V}}(q) = 1 + q^2 + q^3 + 2q^4 + 2q^5 + 4q^6 + 4q^7 + 7q^8 + 8q^9 + 12q^{10} + 14q^{11} + 21q^{12} + 24q^{13} + 34q^{14} + 41q^{15} + 55q^{16} + \cdots \]

(quasi-rational, but not rational)
... the \( W_{23} \) model

- Extend by three fields with \( h=15 \),
- get \( W \) with character

\[
\chi_W(q) = 1 + q^2 + q^3 + 2q^4 + 2q^5 + 4q^6 + 4q^7 + 7q^8 + 8q^9 + 12q^{10} \\
+ 14q^{11} + 21q^{12} + 24q^{13} + 34q^{14} + 44q^{15} + 58q^{16} + \cdots
\]

\[
\chi_V(q) = 1 + q^2 + q^3 + 2q^4 + 2q^5 + 4q^6 + 4q^7 + 7q^8 + 8q^9 + 12q^{10} \\
+ 14q^{11} + 21q^{12} + 24q^{13} + 34q^{14} + 41q^{15} + 55q^{16} + \cdots
\]

- \( W \) is indecomposable but not irreducible

\[
0 \rightarrow W(2) \rightarrow W \rightarrow W(0) \rightarrow 0
\]

- irreducible sub-representation
- irreducible quotient

(Does not happen for the \( W_{1,p} \) models.)
... the $\mathcal{W}_{23}$ model

- 13 irreducibles:

  $\mathcal{W}(h)$ with $h$ from

  (believed to be all)

  $\begin{array}{cccc}
  r = 1 & s = 1 & s = 2 & s = 3 \\
  0, 2, 7 & 0, 1, 5 & \frac{1}{3}, \frac{10}{3} \\
  r = 2 & \frac{5}{8}, \frac{33}{8} & \frac{1}{8}, \frac{21}{8} & -\frac{1}{24}, \frac{35}{24}
  \end{array}$

- These are all self-conjugate, but $\mathcal{W}$ is not $\rightarrow$ new representation $\mathcal{W}^*$

  $$0 \rightarrow \mathcal{W}(2) \rightarrow \mathcal{W} \rightarrow \mathcal{W}(0) \rightarrow 0$$

  $$0 \rightarrow \mathcal{W}(0) \rightarrow \mathcal{W}^* \rightarrow \mathcal{W}(2) \rightarrow 0$$

  (Does not happen for the $\mathcal{W}_{1,p}$ models.)
Fusion of $W_{23}$ representations

- Not known if logarithmic tensor product theory of Huang, Lepowsky, Zhang ’07 applies.  
  (Does not happen for the $W_{1,p}$ models.)

- Compute fusion rules
  - start from representations of $V$
  - compute fusion via Nahm ’94, Gaberdiel, Kausch ’96
    $\rightarrow$ done in Eberle, Flohr ‘06
  - use induced $W$-representations and associativity
  - compare subset to Rasmussen, Pearce ‘08

- 13 irreducibles do not close under fusion, need to add 22 indecomposables to close under fusion + conjugates.
...fusion of $\mathcal{W}_{23}$ representations

• Some fusion rules: $\mathcal{W} \otimes \mathcal{R} = \mathcal{R}$

\[
\mathcal{W}(0) \otimes \mathcal{W}(h) = \begin{cases} 
\mathcal{W}(0) & : h = 0 \\
0 & : \text{else}
\end{cases}
\]

\[
\mathcal{W}(2) \otimes \mathcal{W}(2) = \mathcal{W}^*
\]

• Resolves associativity puzzle in Eberle, Flohr ‘06

\[
\mathcal{W}(0) \otimes (\mathcal{W}(2) \otimes \mathcal{W}(2)) = (\mathcal{W}(0) \otimes \mathcal{W}(2)) \otimes \mathcal{W}(2)
\]

\[
= \mathcal{W}(0) \otimes \mathcal{W}^* = 0
\]

and $\mathcal{W}(0) \otimes \mathcal{W}^* = 0$, while $\mathcal{W}(0) \otimes \mathcal{W} = \mathcal{W}(0)$.
Fusion rules and Grothendieck group $K_0$

Here: $K_0 = \text{equivalence classes } [U] \text{ of representations where } [U] = [V] \text{ iff } \chi_U(q) = \chi_V(q)$

Product on $K_0$ via $[U] \cdot [V] = [U \otimes V]$?

No: Have $\chi_W(q) = \chi_{W^*}(q)$ but

$$[W(0)] \cdot [W] = [W(0)]$$
$$[W(0)] \cdot [W^*] = 0$$

In fact: Tensor product not exact.

(Does not happen for the $W_{1,p}$ models.)
In non-log., rational CFT (Cardy case):

representations $\leftrightarrow$ boundary conditions

boundary changing fields $A \rightarrow B : \mathcal{H}_{A \rightarrow B} = B \otimes A^*$

\[ \begin{array}{c}
A \xrightarrow{\psi} B \end{array} \]

$\mathcal{W}_{2,3}$ model will be the same but only on subset of reps.
(Does not happen for the $\mathcal{W}_{1,p}$ models.)


... boundary theory for the $\mathcal{W}_{2,3}$ model

Problems with $\mathcal{H}_{A \rightarrow B} = B \otimes A^*$:

• **Boundary condition for the irreducible $\mathcal{W}(0)$?**
  \[ \mathcal{H}_{\mathcal{W}(0) \rightarrow \mathcal{W}(h)} = 0 \text{ for } h \neq 0. \]

• **Boundary condition for the vacuum representation $\mathcal{W}$?**
  \[ \mathcal{H}_{\mathcal{W} \rightarrow \mathcal{W}} = \mathcal{W}^* \nRightarrow \mathcal{W} \]
  - no non-degenerate 2-point correlator
  - no embedding of vacuum sector
The rules of the game

Data:
• Labels for boundary conditions $\mathcal{B} = \{A, B, \ldots\}$
• spaces of boundary changing fields $\mathcal{H}_{A \to B}$
• OPE of boundary fields
• boundary one-point correlator on disc $\langle \psi \rangle_A$

Conditions:
• boundary OPE associative
• $\mathcal{H}_{A \to B}$ is non-zero
• $\mathcal{W}$ is a sub-repn of $\mathcal{H}_{A \to A}$ containing vacuum $\Omega$
• boundary two-point correlator $\langle \psi_1 \psi_2 \rangle_A$ is non-deg.
The associativity condition

In non-logarithmic, rational CFT:
boundary OPE coefficients (left out position dependence)

\[ A \cdot B \cdot C = \sum_k C_{ijk}^{ABC} \]

associativity condition

\[ C_{jkq}^{BCD} C_{iql}^{ABD} = \sum_p C_{ijp}^{ABC} C_{pkq}^{ACD} \cdot F_{pq} \]

For \( W_{2,3} \) model:
difficult as conformal 4-point block not known.

Luckily: abstract nonsense construction exists . . .
Interlude: Internal Homs

In a tensor category, an internal Hom $[A,B]$ from $A$ to $B$ is a representing object for the functor $\text{Hom}( - \otimes A , B )$.

$[A,B]$ is an object such that for all $U$:
$$\text{Hom}( U \otimes A , B ) \cong \text{Hom}( U , [A,B] )$$

There is an associative composition
$$m_{C,B,A} : [B,C] \otimes [A,B] \to [A,C]$$

Conjecture: For the $W_{2,3}$ model, $[A,B] = (A \otimes B^*)^*$

(True if $\text{Hom}(U, V^*) \cong \text{Hom}(U \otimes V, \mathcal{W}^*)$ )
the associativity condition

If we set $\mathcal{H}_{A \rightarrow B} = (A \otimes B^*)^*$ then we have an associative boundary OPE for all representations $A, B, \ldots$
The non-degeneracy condition

\[ \mathcal{H}_{A \to B} \times \mathcal{H}_{B \to A} \longrightarrow \mathbb{C} \]

For non-degeneracy need:

\[ \mathcal{H}_{A \to B} \cong (\mathcal{H}_{B \to A})^* \quad \text{i.e.} \quad (A \otimes B^*)^* \cong B \otimes A^* \]

Not true e.g. for \( A=B=W \) as \( W \otimes W^* = W^* \).

A little more abstract nonsense . . .
**Interlude: From duals and conjugates**

A representation $R$ of $W$ has a (right) dual if there is a representation $R^\vee$ and intertwiners

$$d_R : R^\vee \otimes R \rightarrow W \quad \text{and} \quad b_R : W \rightarrow R \otimes R^\vee$$

such that . . .

**Question:** Is the conjugate (contragredient) representation $R^*$ a dual of $R$?

**Answer:** Not always, e.g. $W$ is its own dual, but $W^* \neq W$.

(Does not happen for the $W_{1,p}$ models.)
...boundary theory for the $\mathcal{W}_{2,3}$ model

$\mathcal{B}$ : collection of all $\mathcal{W}$ representations $\mathcal{R}$ for which  
- the conjugate $\mathcal{R}^*$ is a dual of $\mathcal{R}$  
- $b_{\mathcal{R}} : \mathcal{W} \rightarrow \mathcal{R} \otimes \mathcal{R}^*$ is injective

For $A, B \in \mathcal{B}$ have $\mathcal{H}_{A \rightarrow B} = (A \otimes B^*)^* \cong B \otimes A^*$ and one shows:

- boundary OPE associative
- $\mathcal{H}_{A \rightarrow B}$ is non-zero
- $\mathcal{W}$ is a sub-repn of $\mathcal{H}_{A \rightarrow A}$ containing vacuum $\Omega$
- boundary two-point correlator $\langle \psi_1 \psi_2 \rangle_A$ is non-deg.
boundary theory for the $W_{2,3}$ model

- Only 8 of the 13 irreducibles remain:

\[ \mathcal{W}(h) \text{ with } h \text{ from } \]

\[
\begin{array}{c|ccc}
  r = 1 & s = 1 & s = 2 & s = 3 \\
  r = 2 & \frac{5}{8}, \frac{33}{8} & \frac{1}{8}, \frac{21}{8} & -\frac{1}{24}, \frac{35}{24} \\
\end{array}
\]

- Only 18 of the 22 indecomposable generated by fusing irreducibles and taking conjugates remain.

- These are the 26 boundary conditions found in the lattice realisation of Rasmussen, Pearce '08

- NO boundary condition with self-spectrum $W$
Cylinder partition functions

Have $A_{B,C}(q) = \chi_{\mathcal{H}}(q)$ with $\mathcal{H} = \mathcal{H}_{B \rightarrow C}$

Can show:
Multiplication on $K_0$ well defined when restricted to $B$:

$$[B] \cdot [C^*] = [B \otimes C^*]$$

So $A_{B,C}(q)$ only depends on $[B]$ and $[C]$. 
**Torus partition function**

In non-logarithmic, rational CFT:

- **space of bulk states**
- **torus partition function**

\[ \mathcal{H}_{\text{bulk}} = \bigoplus_i U_i \otimes_{\mathbb{C}} U_i^* \]

\[ Z(q) = \sum_i \chi_{U_i}(q) \chi_{U_i}^*(\bar{q}) \]

Denote by \( P_i \rightarrow U_i \) the projective cover of \( U_i \).

In the logarithmic \( \mathcal{W}_{1,p} \) models:

- **space of bulk states**
- **torus partition function**

\[ \mathcal{H}_{\text{bulk}} = \left( \bigoplus_i P_i \otimes_{\mathbb{C}} P_i^* \right) / N \]

\[ Z(q) = \sum_i \chi_{P_i}(q) \chi_{U_i}^*(\bar{q}) \]

*Quella, Schomerus ‘07
Gaberdiel, IR ‘07*
In the $\mathcal{W}_{2,3}$ model:

- $\mathcal{W}(0)$ seems to have no projective cover.
- The sum $\sum_i \chi_{P_i}(q) \chi_{U_i^*}(\bar{q})$ over the remaining 12 irreducibles is not modular invariant.
- Instead

  $$Z(q) = \sum_i \text{dim}(\text{Hom}(P_i, P_i))^{-2} \cdot |\chi_{P_i}(q)|^2$$

  is modular invariant.

- proportional to $Z$ in Feigin, Gainutdinov, Semikhatov, Tipunin '06)
- also correct answer for non-log. Cardy case and $\mathcal{W}_{1,p}$ models.
Summary

• Idea: build boundary theory first, then obtain bulk from boundary.

• Study $W_{2,3}$ model

• Representation theory:
  ($W$ reducible, $W \neq W^*$, $\otimes$ not exact, conjugate $R^*$ vs dual $R^\vee$)

• Boundary theory: $W$-repn $\leftrightarrow$ bnd condition?
  - associative OPE from internal Hom
  - non-deg 2-point correlator guaranteed on subset of reps
  - no boundary condition with self-spectrum $W$