Large-N limit of N=2 theories from localization

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Integrability in Gauge and String Theory, Zürich, 20.08.12

Localization

Pestun'07

- Generic N=2 SYM on S^4

Rem: in a CFT, S^4 is equivalent to R^4

$$Z = \int d^{N-1}a \prod_{i<j} (a_i - a_j)^2 \, \mathrm{e}^{-\frac{8\pi^2}{g^2}\sum_i a_i^2} \mathcal{Z}_{\text{1-loop}}(a) \left| \mathcal{Z}_{\text{inst}}(a; g^2) \right|^2$$

Nekrasov'02

Nekrasov,Okounkov'03

known functions

that depend on field content

$$\langle \Phi \rangle = \mathrm{diag}\,(a_1 \ldots a_n)$$

"vev" of scalar from vector multiplet

$\Downarrow$

integral over Coulomb moduli

Observables

- Free energy:

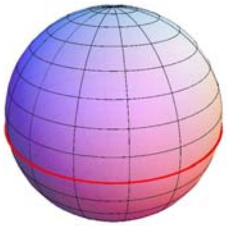
$$F = -\ln Z$$

- Circular Wilson loop:

$$W(C) = \left\langle \sum_{j=1}^{N} \mathrm{e}^{\,2\pi a_j} \right\rangle$$

in field theory defined as:

$$W(C) = \left\langle \mathrm{tr}\,\mathrm{P}\,\exp\left[\int_C ds\,\left(A_\mu(x)\dot{x}^\mu + i\Phi\,|\dot{x}|\right)\right]\right\rangle$$



C

Goal: compute those at N→∞

Rey,Suyama'10
Passerini,Z.'11
Fraser,Prem Kumar'11
Bourgine'11
Passerini,Z.'12

N=4 SYM

Instantons don't contribute, 1-loop corrections cancel

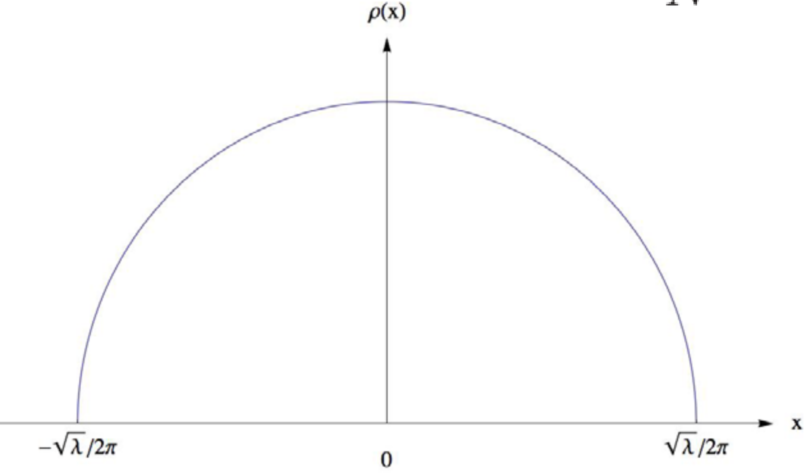
Gaussian matrix model:

$$Z = \int d^{N^2}A \, \mathrm{e}^{-\frac{8\pi^2}{\lambda} N \,\mathrm{tr}\, A^2}$$

Large-N solution:

$$\rho(x) = \frac{1}{N} \mathrm{tr}\, \delta(x - A) = \frac{4}{\lambda}\sqrt{\lambda - 4\pi^2 x^2}$$

Wigner'51



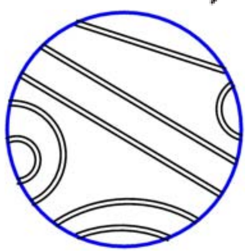
N=4 SYM: Wilson loop

Erickson,Semenoff,Z.'00
Drukker,Gross'00

$$W(C) = \frac{2}{\sqrt{\lambda}} I_1\left(\sqrt{\lambda}\right)$$

$$=$$

$$\sum$$



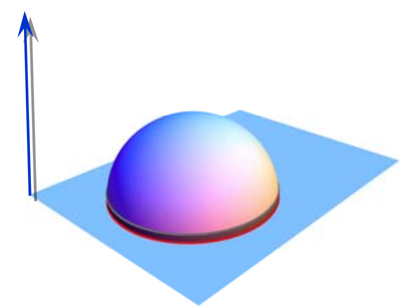
$$\lambda \to \infty$$

$$W(\text{circle}) = \sqrt{\frac{2}{\pi}}\lambda^{-3/4}\,\mathrm{e}^{\sqrt{\lambda}}$$

String fluctuations

Kruczenski,Tirziu'08
Kristjansen,Makeenko'12

Minimal area law in AdS_5

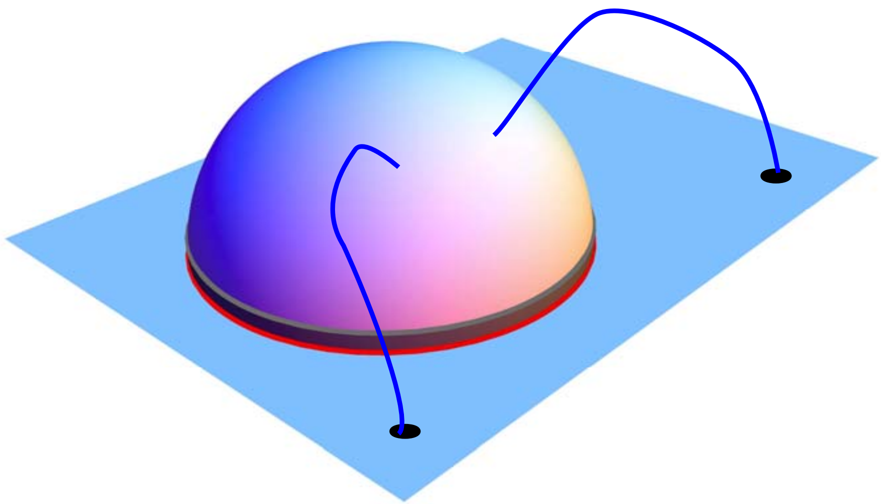


Drukker,Gross,Ooguri'99
Berenstein,Corrado,Fischler,Maldacena'98

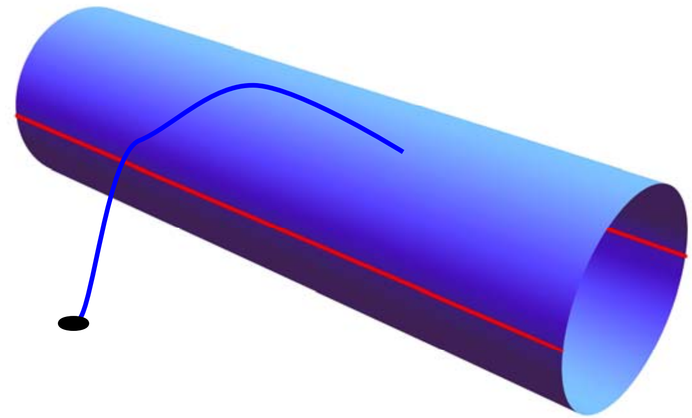
- describes Schwinger pair production

Semenoff,Z.'11
Ambjorn,Makeenko'11

More general correlation functions



$\langle W \mathcal{O} \mathcal{O} \rangle$



$\langle W W \mathcal{O} \rangle$

Giombi,Pestun'12

N=4 SYM: free energy

$$F = -\frac{N^2}{2}\,\ln\lambda$$

According to AdS/CFT:

$$F = -\frac{1}{16\pi G_N}\int_{AdS_5} d^5x\,\sqrt{g}\left(R + \frac{12}{L^2}\right) + I_{\text{boundary}}$$

AdS/CFT dictionary:

$$\frac{\pi L^3}{2G_N} = N^2 \qquad\qquad \frac{L^2}{\alpha'} = \sqrt{\lambda}$$

How does SUGRA know about α’?

Holographic free energy on S^4

Emparan,Johnson,Myers'99

$$ds^2 = \frac{L^2 dr^2}{L^2 + r^2} + r^2 d\Omega_{S^4}^2$$

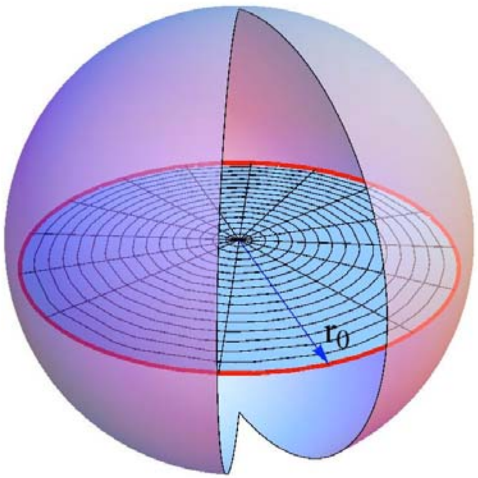
Boundary is S^4 at $r \to \infty$

$$\begin{aligned} F &= -\frac{4\pi^2}{3G_N L} \int_0^{r_0} \frac{dr\, r^4}{\sqrt{L^2 + r^2}} + \text{counterterms} \\ &= \frac{\pi L^3}{2G_N} \ln \frac{r_0}{L} + \ldots \end{aligned}$$

Counterterm that eliminates this log-divergence is also responsible for the holographic Weyl anomaly

Henningson,Skenderis'98
Balasubramanian,Kraus'99

Comparing the cutoffs:



- string action:

$$\frac{L}{2\pi\alpha'}\int_0^{r_0}\frac{dr\,2\pi r}{\sqrt{L^2+r^2}}=\frac{Lr_0}{\alpha'}$$

- W-boson action:

$$2\pi R\Lambda_0$$

field-theory cutoff

$$\boxed{\frac{r_0}{L}=\frac{2\pi R\Lambda_0}{\sqrt{\lambda}}}$$

$$F_{\mathrm{SUGRA}}=-\frac{N^2}{2}\ln\lambda+N^2\ln 2\pi R\Lambda_0$$

If one subtracts the field-theory counterterm, the results agree!

N=2 superCFT

N=2 SYM with $N_f = 2N$

- Unlike N=4 SYM, not integrable

Liendo,Pomoni,Rastelli'11

$$Z = \int d^{N-1}a \prod_{i<j} \left[(a_i - a_j)^2 H^2(a_i - a_j) \right] \mathrm{e}^{-N \sum_i \left(\frac{8\pi^2}{\lambda} a_i^2 + 2 \ln H(a_i) \right)}$$

Pestun'07

$$H(x) \equiv \prod_{n=1}^{\infty} \left(1 + \frac{x^2}{n^2} \right)^n \mathrm{e}^{-\frac{x^2}{n}}$$

- instantons not important at large N

Saddle-point equations

$$\frac{8\pi^2}{\lambda}a_i - K(a_i) - \frac{1}{N}\sum_{j\neq i}\left(\frac{1}{a_i - a_j} - K(a_i - a_j)\right) = 0$$

attraction repulsion repulsion attraction

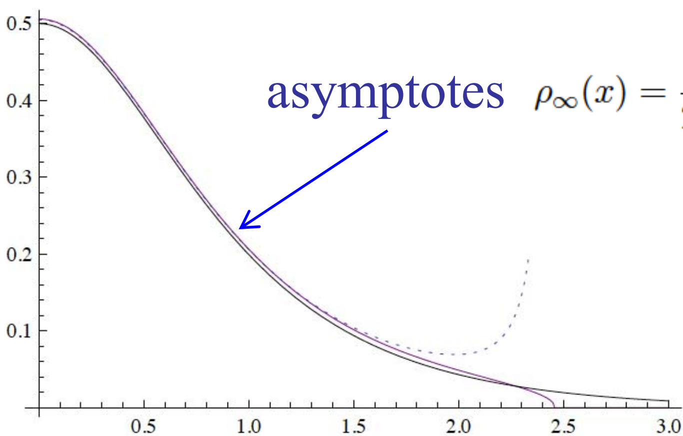
$$K(x) = -\frac{H'(x)}{H} = 2x\sum_{n=1}^{\infty}\left(\frac{1}{n} - \frac{n}{n^2 + x^2}\right) = x\left(\psi(1 + ix) + \psi(1 - ix) - 2\psi(1)\right)$$

$$K(x) \approx 2x\ln x \qquad (x \to +\infty)$$

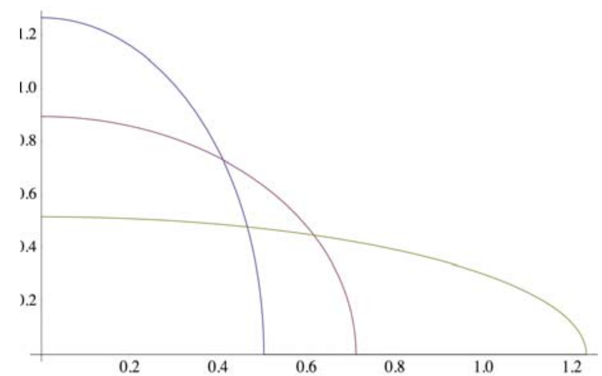
more important at large distances!

$$\int_{-\mu}^{\mu} dy\, \rho(y) \left(\frac{1}{x-y} - K(x-y) \right) = \frac{8\pi^2}{\lambda} x - K(x),$$

$\mu \to \infty$ at $\lambda \to \infty$



N=2



N=4

Free energy

$$\frac{\partial F}{\partial \lambda} = -\frac{8\pi^2 N^2}{\lambda^2} \left\langle a^2 \right\rangle$$

$$\rho_\infty(x) = \frac{1}{2\cosh\frac{\pi x}{2}} \quad \Longrightarrow \quad \left\langle a^2 \right\rangle_\infty = 1$$

$$F = \text{const} + \frac{8\pi^2 N^2}{\lambda}$$

1/λ corrections:

$$F = \text{const} + N^2\left[\frac{8\pi^2}{\lambda} - \left(64\pi^2 \ln^2\lambda - 2^{\frac{7}{2}}\pi^{\frac{1}{2}} b \ln^{\frac{3}{2}}\lambda + O(\ln\lambda)\right)\frac{1}{\lambda^2}\right]$$

Russo,Z.'12

$$b = 0.4018\ldots$$

Holographic interpretation?

Wilson loop

$$\int \frac{dx\ \mathrm{e}^{2\pi x}}{2\cosh\frac{\pi x}{2}}$$ diverges $\Longrightarrow$ need end-point behavior

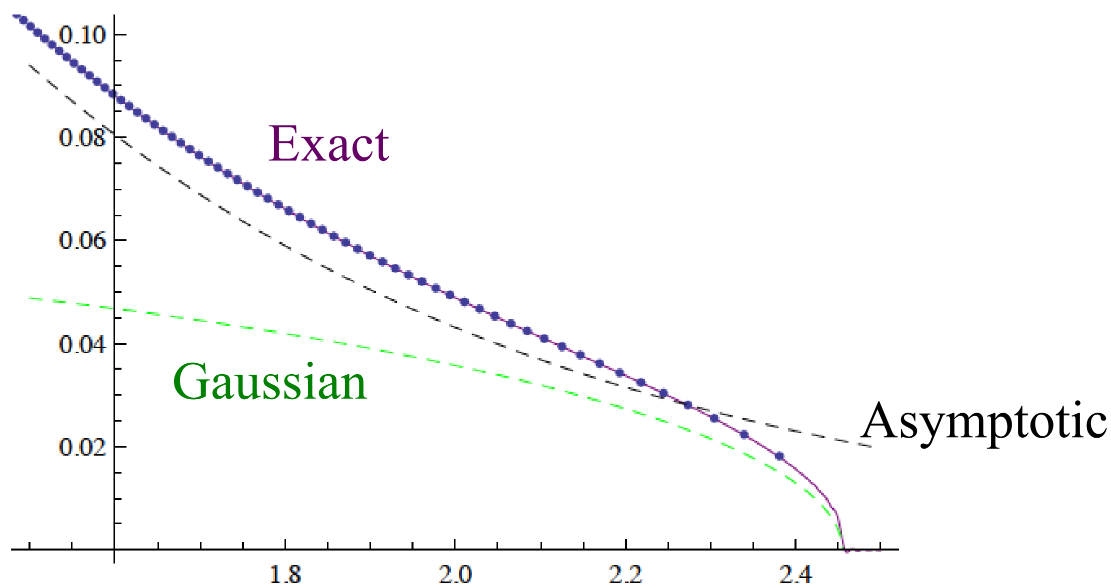
Gaussian matrix model: $$\rho(x) = \frac{8\pi}{\lambda}\sqrt{\mu^2 - x^2} \simeq 8\sqrt{2}\pi\,\frac{\sqrt{\mu}}{\lambda}\sqrt{\mu - x}$$

Exact density:

$$\rho(x) = \frac{\sqrt{\mu}}{\lambda}\, f(\mu - x)$$

$f(z) \sim \sqrt{z}$ at $z \to 0^+$

$f(z) \sim z^{-3/2}$ at $z \to \infty$



Normalization:

$$\int^{\mu} dx\, \rho(x) \sim \frac{\sqrt{\mu}}{\lambda} \int_0^{\infty} dz\, f(z) \sim \frac{\sqrt{\mu}}{\lambda} \quad \text{vs.} \quad \int_{\mu}^{\infty} dx\, \rho_\infty(x) \simeq \frac{2}{\pi}\, \mathrm{e}^{-\pi\mu/2}$$

$$\boxed{C\sqrt{\mu}\,\mathrm{e}^{\pi\mu/2} = \lambda}$$

Wiener-Hopf method: $C \simeq 14.60$

$$W(C) = \text{const}\, \frac{\sqrt{\mu}}{\lambda}\, \mathrm{e}^{2\pi\mu}$$

At $\lambda \to \infty$:

$$W(C_{\rm circle}) = {\rm const}\,\frac{\lambda^3}{(\ln\lambda)^{3/2}}$$

Passerini, Z.'11

$${\rm const} \simeq 9.47\cdot 10^{-5}$$

- quantum-fluctuation prefactor for tensionless string?
- string with effective tension

$$T = \frac{3}{2\pi}\,\ln\lambda \quad ?$$

Pure N=2 SYM

$$Z = \int d^{N-1}a \prod_{i<j} \left[(a_i - a_j)^2 H^2(a_i - a_j) \right] \mathrm{e}^{-\frac{8\pi^2 N}{\lambda} \sum_i a_i^2}$$

Pestun'07

Asymptotically free:

$$\frac{4\pi^2}{\lambda_0} - \ln\left(e^{\gamma} \Lambda_0 R\right) \equiv \frac{4\pi^2}{\lambda}$$

Dynamically generated scale:

$$\Lambda = \Lambda_0 \exp(-4\pi^2/\lambda_0 + \gamma)$$

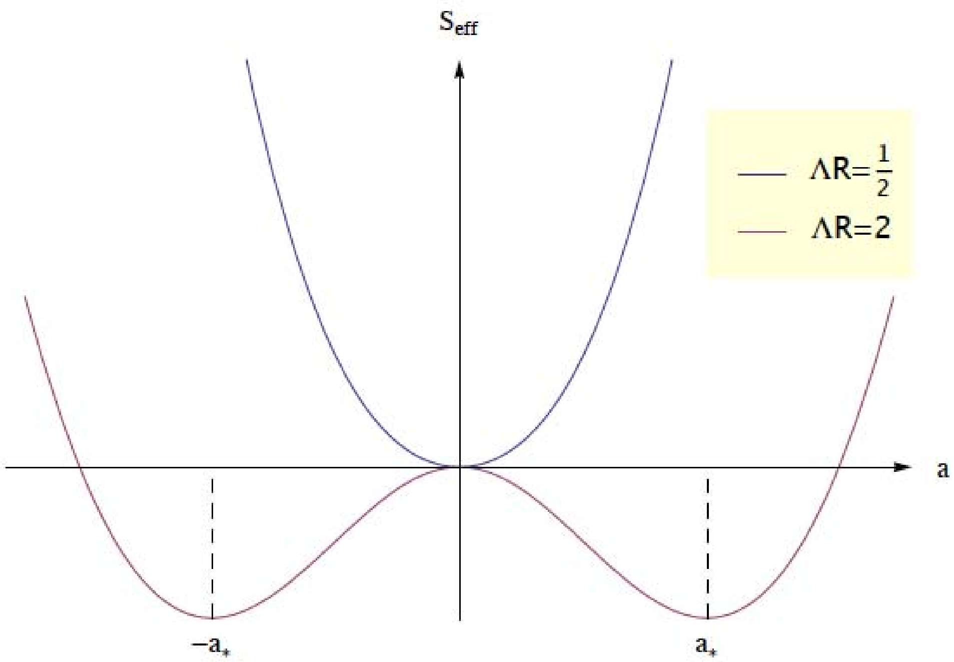
$$\frac{4\pi^2}{\lambda} = -\ln\left(\Lambda R\right)$$

Effective potential

Potential for a single eigenvalue /SU(2)/:

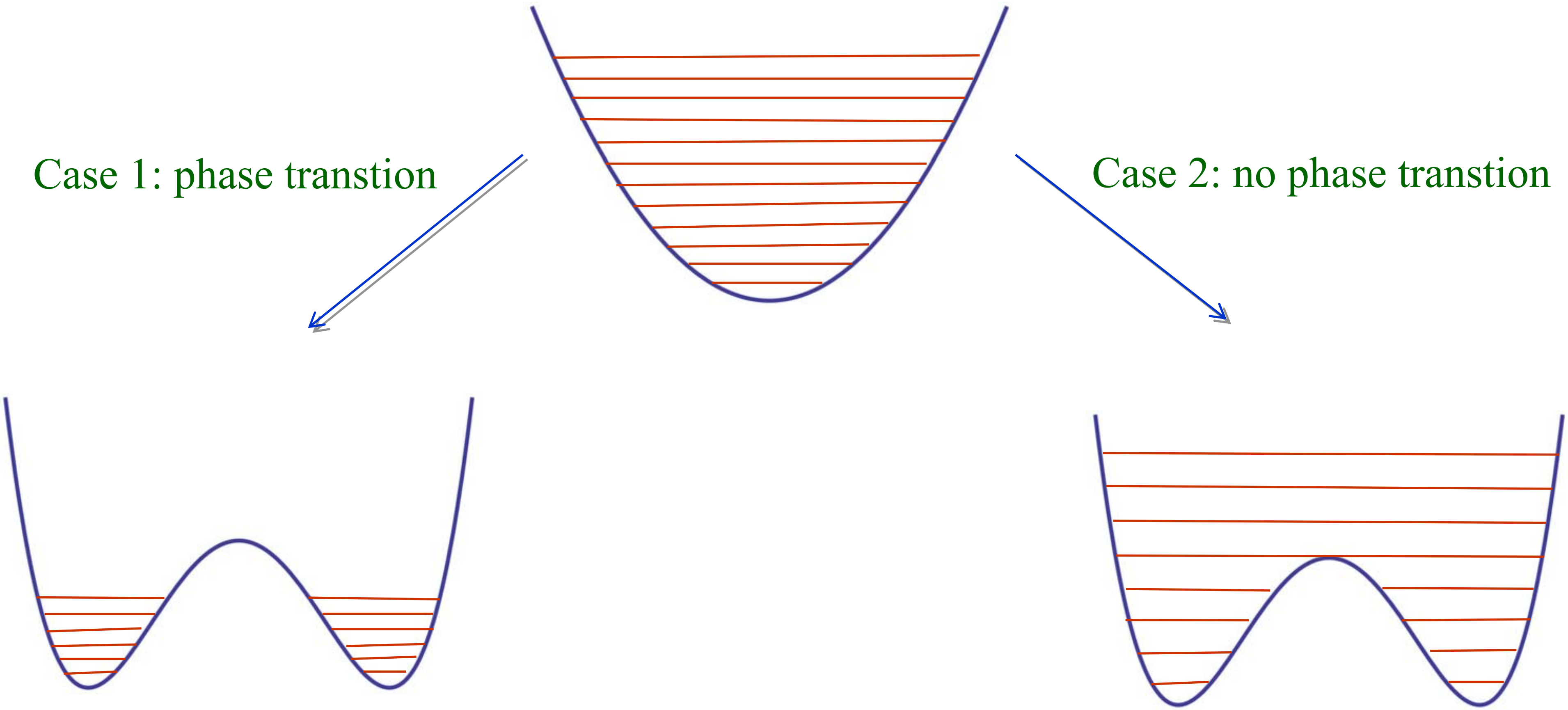
$$S_{\text{eff}}(a) = -8a^2 \ln(\Lambda R) - 2 \ln H(2a)$$

$$\ln H(x) = \begin{cases} -\frac{\zeta(3)}{2} x^4 + O(x^6) & \text{at } x \to 0 \\ -x^2 \ln|x| e^{\gamma - \frac{1}{2}} + O(\ln x) & \text{at } x \to \infty \end{cases}$$



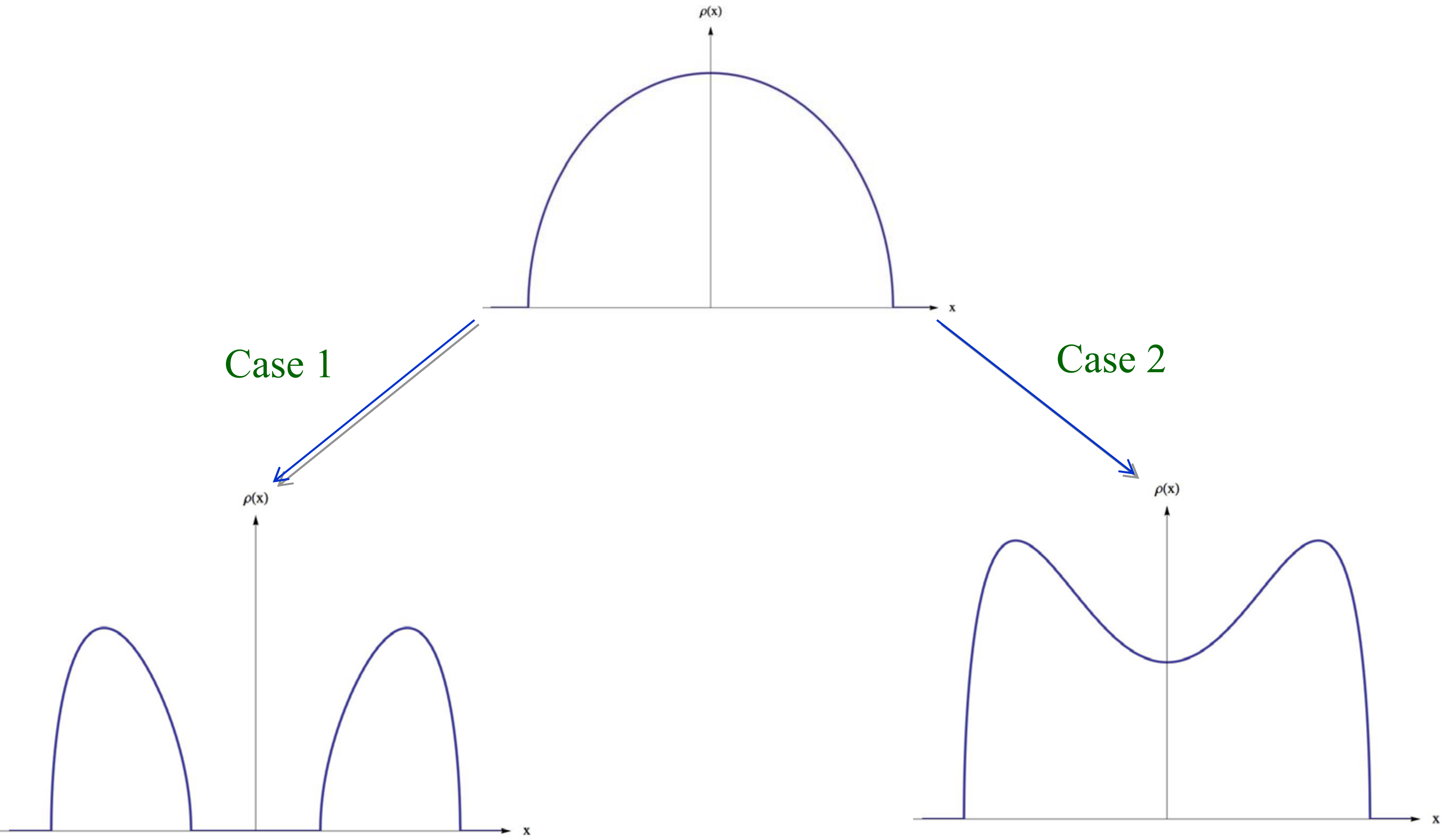
Coleman-Weinberg!

Coleman-Weinberg mechanism at large-N



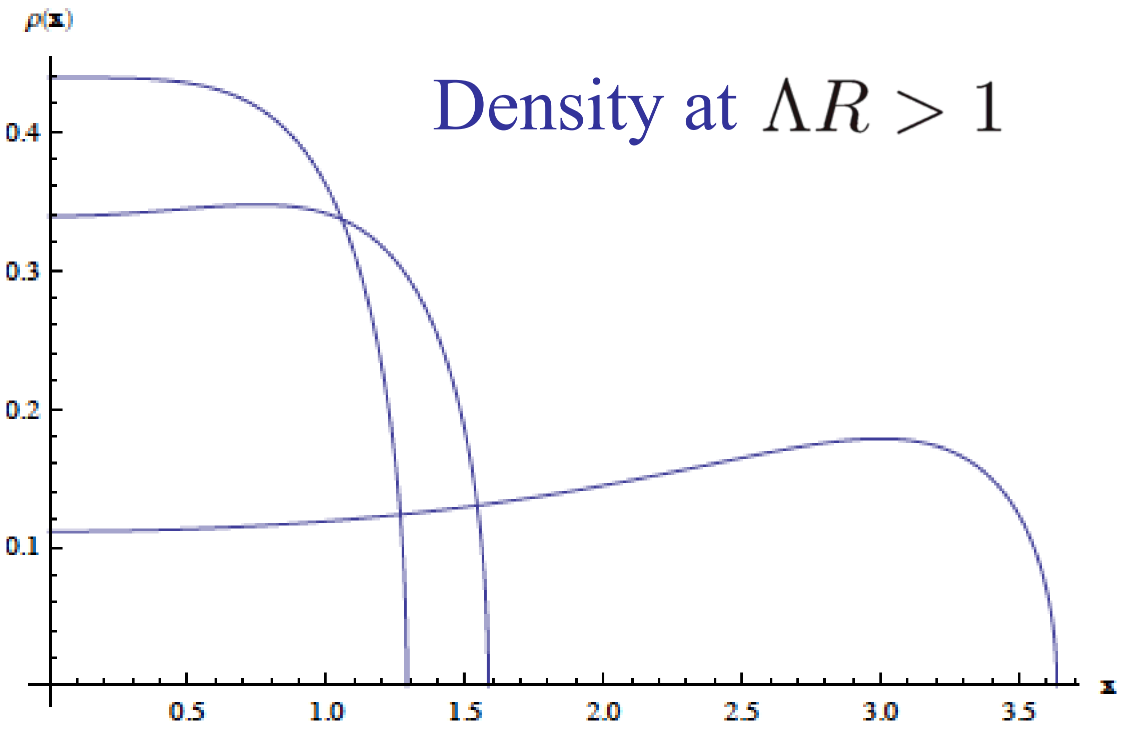
Which case is realized is a dynamical question.

Eigenvalue density



Saddle-point equation

$$\fint_{-\mu}^{\mu} dy\rho(y)\left(\frac{1}{x-y} - K(x-y)\right) = \frac{8\pi^2}{\lambda}x \equiv -2x\ln(\Lambda R)$$



Density at $\Lambda R > 1$

- no phase transition!

Large radius

Eigenvalues spread over large interval

On average, $|a_i - a_j| \gg 1$

$$K(x) \simeq 2x \ln\left(|x|\, \mathrm{e}^{\,\gamma}\right) \gg \frac{1}{x}$$

Eigenvalue density takes on scaling form:

$$\rho(x) = \hat{\rho}(x/\mu)/\mu$$

$$\mu \to \infty$$

Hilbert kernel (1/x) scales away

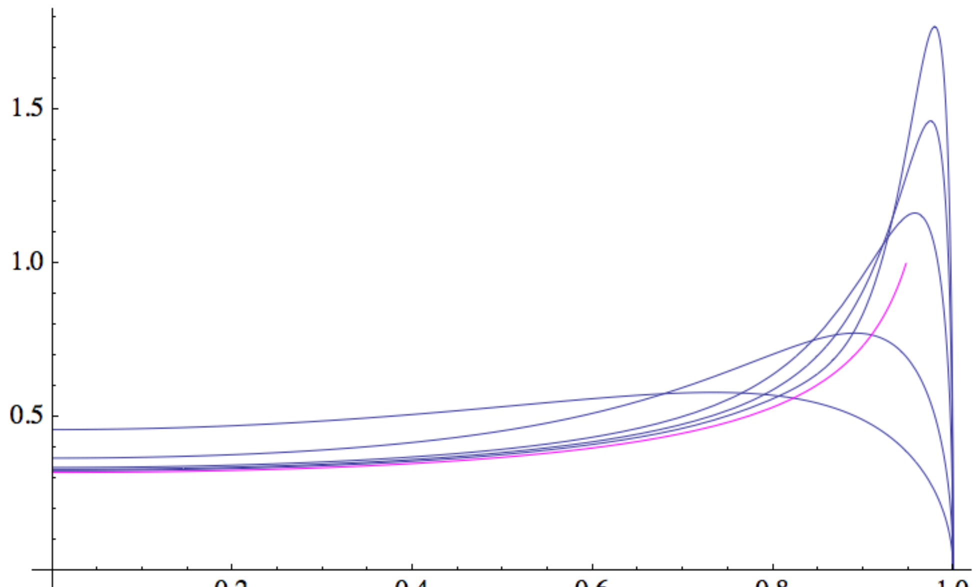
$$\int_{-\mu}^{\mu} dy\,\rho(y)(x-y)\ln\left(|x-y|\,\mathrm{e}^{\gamma}\right) = x\ln\Lambda R$$

$$\Downarrow$$

$$\fint_{-\mu}^{\mu} \frac{dy\,\rho(y)}{x-y} = 0$$

Unique normalizable solution:

$$\rho(x) = \frac{1}{\pi\sqrt{\mu^2 - x^2}}$$



$$\int_{-\mu}^{\mu} dy\, \frac{(x-y)\ln|x-y|}{\pi\sqrt{\mu^2-y^2}} = x \ln\frac{\mu \mathrm{e}}{2}$$

$$\boxed{\mu \simeq 2\,\mathrm{e}^{-1-\gamma}\Lambda R}$$

Wilson loop

$$W(C_{\text{circle}}) = \pi I_0(2\pi\mu) \approx \frac{\text{const}}{\sqrt{\mu}}\,\mathrm{e}^{\,2\pi\mu}$$

$$\boxed{W(C_{\text{circle}}) \sim \mathrm{e}^{\,\text{const}\,\Lambda R}}$$

$$\text{const} = 4\pi\,\mathrm{e}^{\,-1-\gamma}$$

Perimeter law (no confinement!):

finite mass renormalization

for a heavy source

Free energy

$$\frac{\partial F}{\partial \ln R} = -2N^2 \left\langle a^2 \right\rangle$$

$$\left\langle a^2 \right\rangle = \frac{\mu^2}{2}$$

$$\boxed{F = -\,\mathrm{const}\, N^2 \Lambda^2 R^2} \qquad \mathrm{const} = 2\,\mathrm{e}^{-2-2\gamma}$$

Could not be extensive: no cosmological constant

finite renormalization of Newton's constant

N=2*

N=2 SYM with massive adjoint hypermultiplet:
mass deformation of N=4 SYM

$$\int_{-\mu}^{\mu} dy \rho(y) \left(\frac{1}{x-y} - K(x-y) + \frac{1}{2} K(x-y+MR) + \frac{1}{2} K(x-y-MR) \right) = \frac{8\pi^2}{\lambda} x$$

N=4 limit:

$M \to 0$ all K's cancel

Pure N=2 limit:

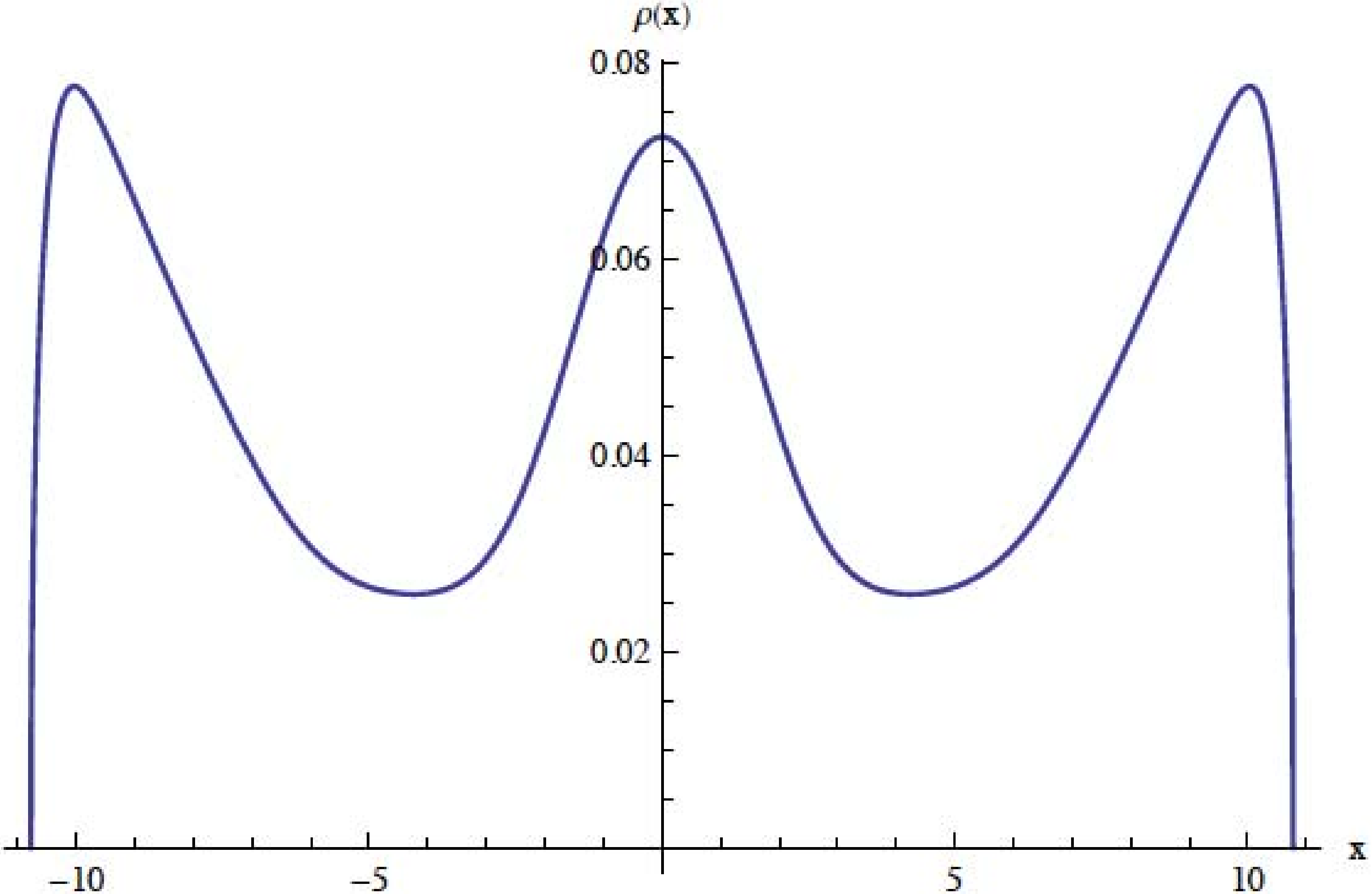
$M \to \infty$ $\qquad K(x \pm MR) = \pm K(MR) \pm K'(MR)x + \ldots$

results in

$$\frac{8\pi^2}{\lambda} \to \frac{8\pi^2}{\lambda_R} = \frac{8\pi^2}{\lambda} - K'(MR) \simeq \frac{8\pi^2}{\lambda} - 4\ln(MR\,\mathrm{e}^{\,1+\gamma})$$

Dynamical scale: $\quad \Lambda = \mathrm{e}^{\,1+\gamma-\frac{4\pi^2}{\lambda}} M$

Intermediate masses/couplings



Conclusions

- Localization/matrix models have been really useful in *verifying* AdS/CFT for maximally supersymmetric theories in D=3,4,5.
- Can localization be used to *derive* AdS/CFT for less supersymmetric theories?
    - ✓ Rem: I think, the answer is "No": too few observables.
- At any rate, the string duals of the less-supersymmetric theories are quite different from N=4 at strong coupling.