The many secrets of AdS/CFT

Alessandro Torrielli

University of Surrey, UK

IGST 2012 Zürich

In collaboration with

$$\begin{pmatrix} \text{G. Arutyunov} & \text{A. Babichenko} & \text{M. de Leeuw} & \text{I. Heckenberger} \\ \text{T. Matsumoto} & \text{S. Moriyama} & \text{I. Plefka} & \text{V. Regelskis} \\ \text{C. Sieg} & \text{F. Spill} & \text{R. Suzuki} & \text{H. Yamane} \end{pmatrix}$$

In progress with

(E. Sklyanin & B. Vicedo   O. Ohlsson Sax & B. Stefański   J. Goodwin)

Reviews $\longrightarrow$ **arXiv: 1012.3982 1012.4005 1104.2474**
with Beisert *et al.* **1204.2366**

Introduction

- Hopf algebras in AdS/CFT
- Non-abelian symmetries and the Yangian

Secret symmetries Spectrum, Amplitudes, Pure spinors, Boundaries, q-deformations

EXACT S-MATRICES

{*for rev*} [P. Dorey '98]

2D integrable massive field theories (worldsheet counterpart)

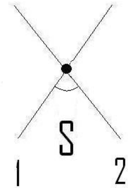
- No particle production/annihilation
- Equality of initial and final sets of momenta
- Factorisation: $S_{M \longrightarrow M} = \prod S_{2 \longrightarrow 2}$

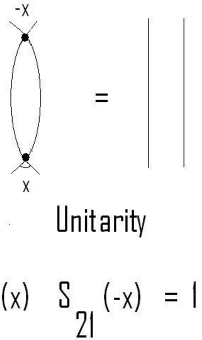
  (all info in 2-body processes)

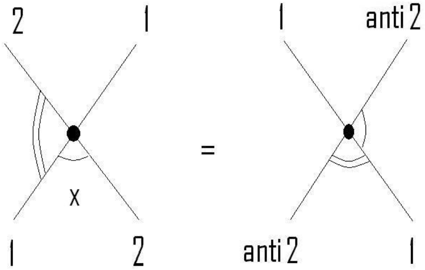
[Staudacher '05; Beisert '05]

Assume relativistic invariance (for now)

$$S_{2\longrightarrow 2} = S(x_1 - x_2) \equiv S(x) \qquad [E_i = m\cosh(x_i),\ p_i = m\sinh(x_i)]$$

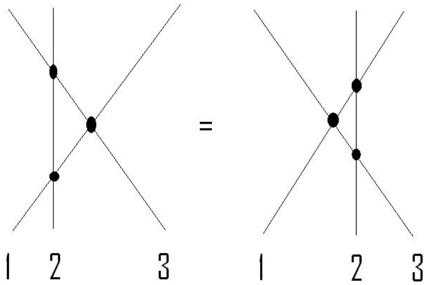




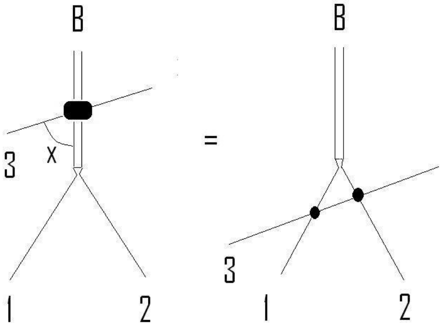


Crossing symmetry $\quad S_{12}(x) = S_{\bar{2}1}(i\pi - x)$

[Janik '06]



Yang-Baxter Equation (YBE) $S_{12}\, S_{13}\, S_{23} \;=\; S_{23}\, S_{13}\, S_{12}$



Bootstrap $S_{3B}(x) = S_{32}\big(x + i(\pi - x_{2B}^{1})\big)\, S_{31}\big(x - i(\pi - x_{1B}^{2})\big)$

[Zamolodchikov-Zamolodchikov '79]

S-MATRIX and HOPF ALGEBRA

{*for rev*} [Delius '95]

Algebraic treatment $\longrightarrow$ relativistic & non (spin chains)

$R : V_1 \otimes V_2 \longrightarrow V_1 \otimes V_2$ R is S-matrix S up to a subtelty which I spare you from

V_i module for a representation of (super)algebra A

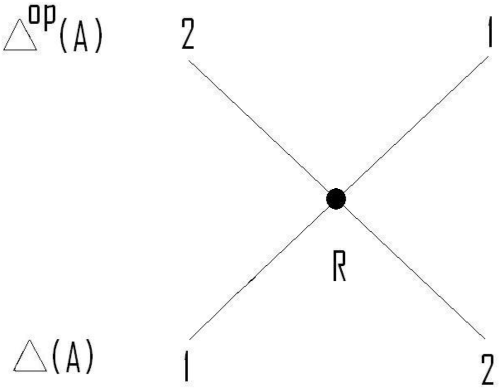
Symmetry on 'in' states: coproduct

$$\Delta : A \longrightarrow A \otimes A$$

$[\Delta(a), \Delta(b)] = \Delta([a, b])$ (homomorphism)

$$(P\Delta)\, R \; = \; R\, \Delta$$

P (graded) permutation $P\Delta$ 'opposite' coproduct Δ^{op} ('out')



Lie (super)algebras can have 'trivial' coproduct

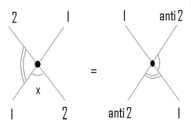
$$\Delta^{op}(Q) = \Delta(Q) = Q \otimes 1 + 1 \otimes Q \qquad \forall\, Q \in A$$

non trivial $\rightarrow$ quantum groups

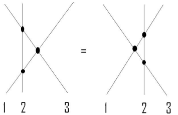
Hopf algebra: coproduct + extra algebraic structures
e.g. antipode (antiparticles) + list of axioms

The Yangian is an ∞-dim non-abelian Hopf algebra

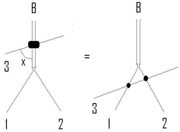
{*books* } [Chari-Pressley '94; Kassel '95; Etingof-Schiffmann '98]



$$(\Sigma \otimes 1)R = R^{-1} = (1 \otimes \Sigma^{-1})R$$



$$R_{12}R_{13}R_{23} = R_{23}R_{13}R_{12}$$



$$(\Delta \otimes 1)R = R_{13}\, R_{23} \qquad (1 \otimes \Delta)R = R_{13}\, R_{12}$$

AdS/CFT

Vacuum $\downarrow = Z = \phi_1 + i\phi_2$

$$\ldots \downarrow \ldots \downarrow \ldots \downarrow \ \Uparrow \ \downarrow \ldots \downarrow \ldots \downarrow \ldots \downarrow \ldots \downarrow \ldots$$

$\mathfrak{psu}(2,2|4) \longrightarrow \mathfrak{psu}(2|2)$ (excitations on the vacuum)

2 bosonic and 2 fermionic d.o.f.; even part $\mathfrak{su}(2) \oplus \mathfrak{su}(2)$

{*rep theory of* $\mathfrak{psu}(2|2)$} [Götz-Quella-Schomerus '05]

Vanishing Killing form $\mathfrak{psl}(n|n)$, $\mathfrak{osp}(2n+2|2n)$, $D(2,1;\alpha)$ $\rightarrow \beta = 0$ [Zarembo '10]

$\mathfrak{psu}(2|2)$ only simple basic classical Lie superalgebra admitting up to 3 central extensions

[Iohara-Koga '01]

AdS/CFT S-matrix: A is centrally-extended $\mathfrak{psu}(2|2)$

[Beisert '05]

$$[\mathbb{L}_a{}^b, \mathbb{J}_c] = \delta_c^b \mathbb{J}_a - \tfrac{1}{2}\delta_a^b \mathbb{J}_c$$
$$[\mathbb{L}_a{}^b, \mathbb{J}^c] = -\delta_a^c \mathbb{J}^b + \tfrac{1}{2}\delta_a^b \mathbb{J}^c$$
$$\{\mathbb{Q}_\alpha{}^a, \mathbb{Q}_\beta{}^b\} = \epsilon_{\alpha\beta}\epsilon^{ab}\ \mathbb{C}$$
$$\{\mathbb{Q}_\alpha^a, \mathbb{G}_b^\beta\} = \delta_b^a \mathbb{R}_\alpha{}^\beta + \delta_\alpha^\beta \mathbb{L}_b{}^a + \tfrac{1}{2}\,\delta_b^a\delta_\alpha^\beta\ \mathbb{H}$$

$$[\mathbb{R}_\alpha{}^\beta, \mathbb{J}_\gamma] = \delta_\gamma^\beta \mathbb{J}_\alpha - \tfrac{1}{2}\delta_\alpha^\beta \mathbb{J}_\gamma$$
$$[\mathbb{R}_\alpha{}^\beta, \mathbb{J}^\gamma] = -\delta_\alpha^\gamma \mathbb{J}^\beta + \tfrac{1}{2}\delta_\alpha^\beta \mathbb{J}^\gamma$$
$$\{\mathbb{G}_a{}^\alpha, \mathbb{G}_b{}^\beta\} = \epsilon^{\alpha\beta}\epsilon_{ab}\ \mathbb{C}^\dagger$$
$$a, b, c = 1, 2 \quad \alpha, \beta, \gamma = 3, 4$$

$$\Phi_\ell \quad = \quad \phi^{a_1\ldots a_\ell} w_{a_1}\ldots w_{a_\ell} + \phi^{a_1\ldots a_{\ell-1}\alpha} w_{a_1}\ldots w_{a_{\ell-1}}\theta_\alpha + \phi^{a_1\ldots a_{\ell-2}\alpha\beta} w_{a_1}\ldots w_{a_{\ell-2}}\theta_\alpha\theta_\beta,$$

$$\mathbb{L}_a{}^b = w_a \frac{\partial}{\partial w_b} - \frac{1}{2}\delta_a^b w_c \frac{\partial}{\partial w_c}$$
$$\mathbb{Q}_\alpha{}^a = \mathrm{a}\,\theta_\alpha \frac{\partial}{\partial w_a} + \mathrm{b}\,\epsilon^{ab}\epsilon_{\alpha\beta} w_b \frac{\partial}{\partial \theta_\beta}$$

$$\mathbb{R}_\alpha{}^\beta = \theta_\alpha \frac{\partial}{\partial \theta_\beta} - \frac{1}{2}\delta_\alpha^\beta \theta_\gamma \frac{\partial}{\partial \theta_\gamma}$$
$$\mathbb{G}_a{}^\alpha = \mathrm{d}\, w_a \frac{\partial}{\partial \theta_\alpha} + \mathrm{c}\,\epsilon_{ab}\epsilon^{\alpha\beta}\theta_\beta \frac{\partial}{\partial w_b}$$

$$\mathbb{C} = \mathrm{ab}\left(w_a \frac{\partial}{\partial w_a} + \theta_\alpha \frac{\partial}{\partial \theta_\alpha}\right) \qquad \mathbb{C}^\dagger = \mathrm{cd}\left(w_a \frac{\partial}{\partial w_a} + \theta_\alpha \frac{\partial}{\partial \theta_\alpha}\right)$$

$$\mathbb{H} = (\mathrm{ad} + \mathrm{bc})\left(w_a \frac{\partial}{\partial w_a} + \theta_\alpha \frac{\partial}{\partial \theta_\alpha}\right)$$

[Arutyunov-Frolov '08]

CHEVALLEY-SERRE [All fermionic Dynkin diagram]

$$\xi_1^+ = -i\nu\,\mathbb{G}_2^4,\quad \xi_2^+ = \mu\,\mathbb{Q}_3^2,\quad \xi_3^+ = \mu\,\mathbb{Q}_4^1,\quad \xi_1^- = -i\mu\,\mathbb{Q}_4^2,\quad \xi_2^- = \nu\,\mathbb{G}_2^3,\quad \xi_3^- = \nu\,\mathbb{G}_1^4$$

$$\mu = iq^{-\frac{p}{2}}\,(q-q^{-1})^{-\frac{1}{2}} \qquad \nu = q^{\frac{p}{2}}\,(q-q^{-1})^{-\frac{1}{2}} \qquad q = \exp\frac{i}{2}$$

[cf. also Gomez-Hernandez '07; Young '07]

$$a_{ij} = \frac{i}{(q-q^{-1})}\begin{pmatrix} 0 & 1 & -1 \\ 1 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix} \qquad \text{Cartan matrix}$$

$$[\kappa_i,\kappa_j] = 0, \quad [\kappa_i,\xi_j^\pm] = \pm\, a_{ij}\,\xi_j^\pm, \qquad i,j = 1,2,3$$

$$[\xi_i^+,\xi_j^-] = \delta_{ij}\,\kappa_j,$$

$$[\xi_i^\pm,[\xi_i^\pm,\xi_j^\pm]] = 0 \qquad \text{for} \quad (i,j) = (1,2),(2,1),(1,3),(3,1)$$

$$[\xi_2^\pm,\xi_3^\pm] = \frac{q^p - q^{-p}}{q-q^{-1}} \qquad [p,\cdot] = 0$$

[NOTE: Serre $\Longrightarrow$ Undeformed is $p \to 0$, *not* $q \to 1$]

Not a mere optical illusion $\to$

HOPF ALGEBRA

Imposing **cocommutative center** [necessary for $\Delta^{op}(C)\,R \;=\; R\,\Delta(C) \;=\; \Delta(C)\,R$]

$$\Delta(p) \;=\; p \otimes 1 \;+\; 1 \otimes p$$

immediately forces

$$\Delta(\kappa_i) \;=\; \kappa_i \otimes 1 \;+\; 1 \otimes \kappa_i$$

$$\Delta(\xi_1^{\pm}) \;=\; \xi_1^{\pm} \otimes q^{\pm\frac{p}{2}} \;+\; q^{\mp\frac{p}{2}} \otimes \xi_1^{\pm}$$
$$\Delta(\xi_i^{\pm}) \;=\; \xi_i^{\pm} \otimes q^{\mp\frac{p}{2}} \;+\; q^{\pm\frac{p}{2}} \otimes \xi_i^{\pm} \qquad i = 2, 3$$

$\longrightarrow$ [Gomez-Hernandez '06; Plefka-Spill-AT '06]

[Klose-McLoughlin-Minahan-Zarembo '06; Arutyunov-Frolov-Plefka-Zamaklar '06]

YANGIANS

$\exists$ Lie (super)algebra Q^A. Suppose $\exists$ additional charges $\hat{Q}^A$

$$[Q^A, Q^B] = i f^{AB}_C \, Q^C \qquad\qquad [Q^A, \hat{Q}^B] = i f^{AB}_C \, \hat{Q}^C$$

(plus Serre) with coproducts

$$\Delta(Q^A) = Q^A \otimes 1 + 1 \otimes Q^A$$
$$\Delta(\hat{Q}^A) = \hat{Q}^A \otimes 1 + 1 \otimes \hat{Q}^A + \frac{i}{2} \, f^A_{BC} Q^B \otimes Q^C$$

[Drinfeld '86; {*rev*} MacKay'04]

(Infinite) Spin-Chain Dolan-Nappi-Witten '03; Agarwal-Rajeev '04; Zwiebel '06; Beisert-Zwiebel '07]

Classical String [Mandal-Suryanarayana-Wadia '02; Bena-Polchinski-Roiban '03; Hatsuda-Yoshida '04; Das-Maharana-Melikyan-Sato '04]

S-matrix Yangian [Beisert '07]

We know after rescaling (and $q = \exp \frac{i}{2}$)

$$\Delta(Q^A) = Q^A \otimes q^{[[A]]\frac{p}{2}} + q^{-[[A]]\frac{p}{2}} \otimes Q^A$$

Correspondingly, $\exists$ centrally-extended $\mathfrak{psu}(2|2)$ Yangian

$$\Delta(\hat{Q}^A) = \hat{Q}^A \otimes q^{[[A]]\frac{p}{2}} + q^{-[[A]]\frac{p}{2}} \otimes \hat{Q}^A + \frac{i}{2} f^A_{BC}\, Q^B\, q^{-[[C]]\frac{p}{2}} \otimes q^{[[B]]\frac{p}{2}}\, Q^C$$

SIGMA MODEL PICTURE

[$\longrightarrow$ Pure spinors]

2D classical field theory, local currents

$$J_\mu = J_\mu^A \, T_A \qquad \partial^\mu J_\mu^A = 0 \qquad Q^A = \int_{-\infty}^{\infty} dx \, J_0^A$$

Flatness (Lax pair)

$$\partial_0 J_1 - \partial_1 J_0 + [J_0, J_1] = 0$$

Non-local conserved current

$$\hat{J}_\mu^A(x) = \epsilon_{\mu\nu} \, \eta^{\nu\rho} \, J_\rho^A(x) + \frac{i}{2} f_{BC}^A \, J_\mu^B(x) \int_{-\infty}^{x} dx' \, J_0^C(x')$$

$$\frac{d}{dt} \hat{Q}^A = \frac{d}{dt} \int_{-\infty}^{\infty} dx \, \hat{J}_0^A(x) = 0$$

Prototype: Principal Chiral Model

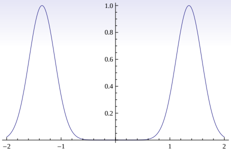
$$L = Tr[\partial_\mu g^{-1} \partial^\mu g] \qquad g \in Lie$$

(left,right) global symmetry $\qquad g \longrightarrow e^{i\lambda} g,\; g\, e^{i\lambda}$

Flat currents $\qquad J^{L,R} = (\partial_\mu g) g^{-1},\; g^{-1} (\partial_\mu g) \in lie$

String $\longrightarrow \frac{PSU(2,2|4)}{SO(4,1)\times SO(5)}$ coset [Metsaev-Tseytlin action]

$\longrightarrow$ pure spinor action [Berkovits]



Classical argument [Lüscher-Pohlmeyer '78; MacKay '92]:

$$\hat{Q}^A = \int_{-\infty}^{\infty} dx\, J_1^A(x) + \frac{i}{2} f_{BC}^A \int_{-\infty}^{\infty} dx\, J_0^B(x) \int_{-\infty}^{x} dx'\, J_0^C(x')$$

Evaluating on profile

$$\hat{Q}^A_{|profile} = \int_{-\infty}^{0} J_1^A + \frac{i}{2} f_{BC}^A \int_{-\infty}^{0} J_0^B \int_{-\infty}^{x} J_0^C$$

$$+ \int_{0}^{\infty} J_1^A + \frac{i}{2} f_{BC}^A \int_{0}^{\infty} J_0^B \int_{0}^{x} J_0^C + \frac{i}{2} f_{BC}^A \int_{0}^{\infty} J_0^B \int_{-\infty}^{0} J_0^C$$

$$\longrightarrow \Delta(\hat{Q}^A) = \hat{Q}^A \otimes 1 + 1 \otimes \hat{Q}^A + \frac{i}{2} f_{BC}^A\, Q^B \otimes Q^C$$

REMARKS

- Evaluation representation:

$$\hat{Q}^A = u\, Q^A = ig\,(x^+ + \frac{1}{x^+} - \frac{i}{2g})\, Q^A$$

$x^\pm = x^\pm(p)$ also parameterize algebra representation

- Yangian symmetry in evaluation rep $\rightarrow$ *difference form*

$$S = S(u_1 - u_2)$$

S-matrix does not to possess this symmetry because $u = u(x^\pm)$

(see however [AT '09])

- Coproduct needs raising index f_C^{AB} with inverse Killing form, but...

...for $\mathfrak{psu}(2|2)$, this does not exist, yet table of coproducts can be fully determined (cf. extension by automorphisms

[Spill 'dipl.thesis, Beisert '06])

- For higher bound-states, either YBE or Yangian symmetry have to be used to completely fix S-matrix

[Arutyunov-Frolov '08, de Leeuw '08, MacKay-Regelskis '10]

Drinfeld's second realization

[Drinfeld '88]

$$[\kappa_{i,m}, \kappa_{j,n}] = 0, \quad [\kappa_{i,0}, \xi^{\pm}_{j,m}] = \pm a_{ij}\, \xi^{\pm}_{j,m},$$
$$[\xi^{+}_{i,m}, \xi^{-}_{j,n}] = \delta_{ij}\, \kappa_{j,m+n},$$

$$[\kappa_{i,m+1}, \xi^{\pm}_{j,n}] - [\kappa_{i,m}, \xi^{\pm}_{j,n+1}] = \pm\frac{1}{2} a_{ij} \{\kappa_{i,m}, \xi^{\pm}_{j,n}\},$$

a_{ij} Cartan matrix

$$[\xi^{\pm}_{i,m+1}, \xi^{\pm}_{j,n}] - [\xi^{\pm}_{i,m}, \xi^{\pm}_{j,n+1}] = \pm\frac{1}{2} a_{ij} \{\xi^{\pm}_{i,m}, \xi^{\pm}_{j,n}\},$$

$$n_{ij} = 1 + |a_{ij}| \quad Sym_{\{k\}} [\xi^{\pm}_{i,k_1}, [\xi^{\pm}_{i,k_2}, \dots [\xi^{\pm}_{i,k_{n_{ij}}}, \xi^{\pm}_{j,l}] \dots]] = 0$$

Serre relations

Mechanical generation of higher levels

$$[\kappa_{i,m}, \kappa_{j,n}] = 0, \qquad [\kappa_{i,0}, \xi^{\pm}_{j,m}] = \pm a_{ij}\, \xi^{\pm}_{j,m},$$

$\kappa_{i,m}$ generate Cartan subalgebra

$$[\kappa_{i,m+1}, \xi^{\pm}_{j,n}] - [\kappa_{i,m}, \xi^{\pm}_{j,n+1}] = \pm\frac{1}{2} a_{ij} \{\kappa_{i,m}, \xi^{\pm}_{j,n}\}, \quad a_{ij} \text{ Cartan matrix}$$

Suppose you know full level 0 and 1 rep. Choose i, j s.t. $a_{ij} \neq 0$

$$[\kappa_{i,0+1}, \xi^{\pm}_{j,1}] - [\kappa_{i,0}, \xi^{\pm}_{j,1+1}] = \pm\frac{1}{2} a_{ij} \{\kappa_{i,0}, \xi^{\pm}_{j,1}\} =$$

$$[\kappa_{i,1}, \xi^{\pm}_{j,1}] \mp a_{ij}\, \xi^{\pm}_{j,2} = \pm\frac{1}{2} a_{ij} \{\kappa_{i,0}, \xi^{\pm}_{j,1}\}$$

Level 2 is explicit (undoing of a Lie bracket)

AdS/CFT has central extension

[Spill-AT '08]

$$[\kappa_{i,m}, \kappa_{j,n}] = 0 \quad [\kappa_{i,0}, \xi^{\pm}_{j,m}] = \pm a_{ij}\, \xi^{\pm}_{j,m}$$
$$[\xi^{+}_{i,m}, \xi^{-}_{j,n}] = \delta_{ij}\, \kappa_{j,m+n},$$

$$b_{ij} = \begin{pmatrix} 0 & 1 & -1 \\ 1 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix} = -i(q - q^{-1})\, a_{ij} \equiv \hbar\, a_{ij} \qquad \hbar = 2\sin(\tfrac{1}{2})$$

$$[\kappa_{i,m+1}, \xi^{\pm}_{j,n}] - [\kappa_{i,m}, \xi^{\pm}_{j,n+1}] = \pm\frac{\hbar}{2} a_{ij}\, \{\kappa_{i,m}, \xi^{\pm}_{j,n}\}$$
$$[\xi^{\pm}_{i,m+1}, \xi^{\pm}_{j,n}] - [\xi^{\pm}_{i,m}, \xi^{\pm}_{j,n+1}] = \pm\frac{\hbar}{2} a_{ij}\, \{\xi^{\pm}_{i,m}, \xi^{\pm}_{j,n}\}$$

$$(i,j) = (1,2), (2,1), (1,3), (3,1) \qquad Sym_{\{k\}}[\xi^{\pm}_{i,k_1}, [\xi^{\pm}_{i,k_2}, \xi^{\pm}_{j,l}]] = 0$$

$$[\xi^{\pm}_{2,m}, \xi^{\pm}_{3,n}] = C^{\pm}_{m+n}, \qquad [C^{\pm}_{n}, \cdot] = 0 \qquad C^{\pm}_{0} = \frac{q^{p} - q^{-p}}{q - q^{-1}}$$

UNIVERSAL R-MATRIX

Given H non *co*-commutative Hopf algebra ($\Delta^{op} \neq \Delta$),
suppose $\exists$ abstract solution $R \in \mathrm{H} \otimes \mathrm{H}$ of

$$\Delta^{op} R = R\, \Delta$$

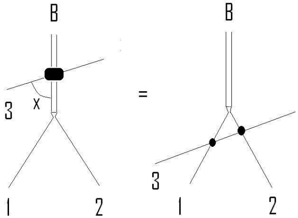
Universal: independent of reps in each factors of $\otimes$

Standard Yangian is one such H

Theorem (Drinfeld): if R satisfies Quasi-Triangularity

(rep-independent version of bootstrap principle)

$$(\Delta \otimes 1)R = R_{13}\, R_{23}$$
$$(1 \otimes \Delta)R = R_{13}\, R_{12}$$



then it also satisfies YBE and Crossing

Direct proof of properties of the S-matrix

Complete solution to scattering problem reduces to:

find the abstract tensor R given H, and then project it into your favorite (bound-state) reps

Define an integrable model and its correlation functions purely *via* its quantum group symmetry

[LeClair-Smirnov '92]

Q-operator [Bazhanov,Frassek,Łukowski,Meneghelli,Staudacher]

Can we do it for AdS/CFT?

Yangian in short reps relatively well understood, *e.g.* all bound state S- and transfer matrices [Arutyunov-de Leeuw-(+Suzuki)-AT '09]

Classical r-matrix [AT '07, Moriyama-AT '07, Beisert-Spill '07] has nice classical double structure, but no quantization yet

Long reps show the need of extending the algebra, or else universal R-matrix does not $\exists$ [Arutyunov-de Leeuw-AT '10]

$\longrightarrow$ SECRET YANGIAN SYMMETRY [Matsumoto-Moriyama-AT '07, Beisert-Spill '07] with $\mathfrak{gl}(2|2)$ signature may be the key

$$\Delta(\hat{B}) = \hat{B} \otimes 1 + 1 \otimes \hat{B} + \frac{i}{2g}(\mathbb{G}_a^\alpha \otimes \mathbb{Q}_\alpha^a + \mathbb{Q}_\alpha^a \otimes \mathbb{G}_a^\alpha)$$

$$\Sigma(\hat{B}) = -\hat{B} + \frac{2i}{g}\mathbb{H}$$

$$\hat{B} = \frac{1}{4}(x^+ + x^- - 1/x^+ - 1/x^-)\, diag(1,1,-1,-1)$$

[Exact in g, not a strong coupling exp.]

Universal R-matrix

[Khoroshkin-Tolstoy '96; Rej-Spill '11]

should look smtg. like $R = R_E\, R_H\, R_F$

$$R_H = \exp\left\{ \mathrm{Res}_{u=v} \left[\sum_{i,j} \frac{\mathrm{d}}{\mathrm{d}u} (\log H_i^+(u)) \otimes D_{ij}^{-1} \log H_j^-(v) \right] \right\}$$

$D_{ij} = -(T^{\frac{1}{2}} - T^{-\frac{1}{2}})\, a_{ij}(T^{\frac{1}{2}})$, $a_{ij}(t) = \frac{t^{a_{ij}} - t^{-a_{ij}}}{t - t^{-1}}$, $Tf(u) = f(u+1)$

$$\mathrm{Res}_{u=v}\,(A(u) \otimes B(v)) = \sum_k a_k \otimes b_{-k-1}$$

for $A(u) = \sum_k a_k u^{-k-1}$ and $B(u) = \sum_k b_k u^{-k-1}$

$$H_i^{\pm}(u) = 1 \pm \sum_{\substack{n \geq 0 \\ n < 0}} h_n u^{-n-1}$$

Problem

$$R \sim \prod_{\alpha \in \Delta_+} \exp\left[e_\alpha \otimes e_{-\alpha}\right] \cdot \exp\left[a_{ij}^{-1} h^i \otimes h^j\right] \cdot \cdot \prod_{\alpha \in \Delta_+} \exp\left[e_{-\alpha} \otimes e_\alpha\right]$$

(Centrally extended) $\mathfrak{psl}(2|2)$ has degenerate Cartan matrix

Khoroshkin-Tolstoy: go to the next non-degenerate one ($\mathfrak{gl}(2|2)$)

$\mathfrak{gl}(2|2)$ contains B which makes a_{ij} non-degenerate

More problems

The Yangian cannot be $Y(\mathfrak{gl}(2|2))$ since at level zero

$$\Delta(B) = B \otimes 1 + 1 \otimes B \qquad\qquad B \propto diag(1, 1, -1, -1)$$

is not a symmetry, *e.g* $\quad R\,|\phi_1\rangle \otimes |\phi_2\rangle \propto \ldots + |\psi_3\rangle \otimes |\psi_4\rangle$

It starts at level one

Indentation: Perhaps new type of quantum groups

[Etingof, priv comm]

SECRET (BONUS) SYMMETRY IN AMPLITUDES

Tree-level planar n-particle color-ordered amplitudes

$\mathcal{A}_n(\lambda_k, \tilde{\lambda}_k, \eta_k) \quad k = 1, \ldots, n$ Null mom. $p_k^\mu \sigma_\mu^{\alpha\dot{\beta}} = \lambda_k^\alpha \tilde{\lambda}_k^{\dot{\beta}}$

$\lambda_k, \tilde{\lambda}_k \in \mathbb{C}^2$ c.c. spinors $\eta_k \in \mathbb{C}^{0|4}$ Grassmann

Yangian of $\mathfrak{psu}(2,2|4)$ [Drummond-Henn-Plefka '09; cf. He's talk]

$$\mathfrak{J}^A = \sum_{i=1}^{n} \mathfrak{J}_i^A \qquad \widehat{\mathfrak{J}}^A = f^A_{BC} \sum_{j<k=1}^{n} \mathfrak{J}_j^B \mathfrak{J}_k^C$$

$$\widehat{\mathfrak{B}} = \sum_{k=1}^{n-1} \sum_{j=k+1}^{n} \left(\mathfrak{Q}_k^{\alpha b} \mathfrak{S}_{j,\alpha b} - \bar{\mathfrak{Q}}_{k,b}^{\dot{\alpha}} \bar{\mathfrak{S}}_{j,\dot{\alpha}}^{b} - \mathfrak{Q}_j^{\alpha b} \mathfrak{S}_{k,\alpha b} + \bar{\mathfrak{Q}}_{j,b}^{\dot{\alpha}} \bar{\mathfrak{S}}_{k,\dot{\alpha}}^{b} \right)$$

[Beisert-Schwab '11]

SECRET (BONUS) SYMMETRY IN PURE SPINORS

Group variable $g \in PSU(2,2|4)$, conserved currents j

$$J = -dg\, g^{-1} \quad \rightarrow \quad J(z) \quad \rightarrow \quad j = g^{-1} \left.\frac{dJ}{d\log z}\right|_{z=1} g$$

Lax connection

$$[\partial_+ + J_+(z)\,,\, \partial_- + J_-(z)] = 0$$

Non-local conserved charges generated by transfer matrix

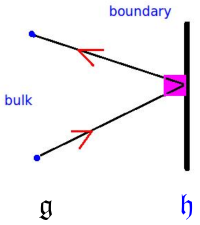
$$T(z) = g(+\infty)^{-1} \left[P\ \exp \int_C \left(-J_+(z) d\tau^+ - J_-(z) d\tau^- \right) \right] g(-\infty)$$

$$I_\xi = \mathrm{Str}\ \xi \left(\int \int_{\sigma_1 > \sigma_2} [j(\sigma_1), j(\sigma_2)] - \int k \right) \qquad \xi \in \mathfrak{psu}(2,2|4)$$

[Berkovits-Mikhailov '11]

SECRET SYMMETRY IN BOUNDARY PROBLEMS

[Regelskis '11] [MacKay-Regelskis '10 -'12]



$\sigma(\mathfrak{h}) = \mathfrak{h}$ $\quad$ $\sigma(\mathfrak{m}) = -\mathfrak{m}$ $\quad$ involution

$$K \ : \ V_1(u) \otimes V_b \longrightarrow V_1(-u) \otimes V_b$$

$[\mathfrak{h}, \mathfrak{h}] \subset \mathfrak{h}$ $\quad$ $[\mathfrak{h}, \mathfrak{m}] \subset \mathfrak{m}$ $\quad$ $[\mathfrak{m}, \mathfrak{m}] \subset \mathfrak{h}$ $\quad$ $\mathfrak{m} = \mathfrak{g}/\mathfrak{h}$

Twisted Yangian $\quad$ $\Delta(\mathfrak{J}) \in Y(\mathfrak{g}) \otimes Y(\mathfrak{g}, \mathfrak{h})$

$Y(\mathfrak{g}, \mathfrak{h})$ gen. by $\mathfrak{J}^i$ and $\tilde{\mathfrak{J}}^p = \widehat{\mathfrak{J}}^p + f^p_{qj} \mathfrak{J}^q \mathfrak{J}^j$ $\quad$ $i \sim \mathfrak{h}$ $\quad$ $p \sim \mathfrak{m}$

[Delius-MacKay-Short '01]

Secret $\tilde{B}$ is there if $\mathfrak{h}$ has degenerate Cartan matrix (but some secret susys always there...)

QUANTUM AFFINE SECRET SYMMETRY

[Beisert-Koroteev '08; De Leeuw-Matsumoto-Regelskis '11] [de Leeuw-Regelskis-AT '12]

$$DA = \begin{pmatrix} 2 & -1 & 0 & -1 \\ -1 & 0 & 1 & 0 \\ 0 & 1 & -2 & 1 \\ -1 & 0 & 1 & 0 \end{pmatrix} \qquad D = \mathrm{diag}(1,-1,-1,-1)$$

$$K_i E_j = q^{DA_{ij}} E_j K_i \ , \qquad K_i F_j = q^{-DA_{ij}} F_j K_i \ ,$$

$$[E_j, F_j\} = D_{jj} \frac{K_j - K_j^{-1}}{q - q^{-1}} \ , \qquad [E_i, F_j\} = 0$$

$$\{[E_1, E_k], [E_3, E_k]\} - (q - 2 + q^{-1}) E_k E_1 E_3 E_k = g\alpha_k (1 - V_k^2 U_k^2) \ ,$$

$$\{[F_1, F_k], [F_3, F_k]\} - (q - 2 + q^{-1}) F_k F_1 F_3 F_k = g\alpha_k^{-1} (V_k^{-2} - U_k^{-2})$$

$$B_{E,F} = b_{E,F}\, diag(1, \ldots, -1 \ldots)$$

$$\begin{aligned}\Delta B_E =& B_E \otimes 1 + 1 \otimes B_E \\ &+ (U_2^{-1} \otimes 1)(K_{123}^{-1} E_4 \otimes \tilde{E}_{123} + K_{23}^{-1} \tilde{E}_{14} \otimes \tilde{E}_{23} + K_{12}^{-1} \tilde{E}_{34} \otimes \tilde{E}_{12} + K_2^{-1} \tilde{E}_{134} \otimes E_2) \\ &+ K_{124}^{-1} E_3 \otimes \tilde{E}_{124} + K_3^{-1} \tilde{E}_{124} \otimes E_3\end{aligned}$$

Drinfeld's second realization of $U_q(\mathfrak{gl}(2|2))$ (all-fermionic Dynkin diagram)

$$[h_{j,m}\,,\,h_{j',n}]=0 \qquad [h_{i,0}\,,\,\xi^{\pm}_{j,m}]=\pm a_{ij}\,\xi^{\pm}_{j,m}$$

$$[h_{i,n}\,,\,\xi^{\pm}_{j,m}]=\pm\frac{[a_{ij}\,n]_q}{n}\,\xi^{\pm}_{j,n+m}\,, \quad n\neq 0 \qquad [x]_q=\frac{q^x-q^{-x}}{q-q^{-1}}$$

$$\{\xi^{+}_{i,n}\,,\,\xi^{-}_{i',m}\}=\frac{\delta_{i,i'}}{q-q^{-1}}\left(\psi^{+}_{i,n+m}-\psi^{-}_{i,n+m}\right)$$

$$\{\xi^{\pm}_{i,m}\,,\,\xi^{\pm}_{i',n}\}=0, \quad \text{if} \quad a_{ii'}=0\,,$$

$$\{\xi^{\pm}_{i,m+1}\,,\,\xi^{\pm}_{i',n}\}_{q^{\pm a_{ii'}}}=\{\xi^{\pm}_{i',n+1},\xi^{\pm}_{i,m}\}_{q^{\pm a_{ii'}}}$$

$$[\{\xi^{\pm}_{2,m}\,,\,\xi^{\pm}_{1,n}\}_q,\{\xi^{\pm}_{2,p}\;\xi^{\pm}_{3,r}\}_{q^{-1}}]=[\{\xi^{\pm}_{2,p}\,,\,\xi^{\pm}_{1,n}\}_q,\{\xi^{\pm}_{2,m}\,,\,\xi^{\pm}_{3,r}\}_{q^{-1}}]$$

$$a_{ij}=\begin{pmatrix} 0 & 1 & 0 & 2\\ 1 & 0 & -1 & -2\\ 0 & -1 & 0 & 2\\ 2 & -2 & 2 & 0\end{pmatrix}$$

$$\psi^{\pm}_i(z)=q^{\pm h_{i,0}}\exp\left(\pm(q-q^{-1})\sum_{m>0}h_{i,\pm m}z^{\mp m}\right)=\sum_{n\in\mathbb{Z}}\psi^{\pm}_{i,n}z^{-n}$$

CONCLUSIONS

- A deep mathematical structure is there, in some aspects almost reducible to standard, in others much harder. Generalizations (cf. Yoshida's talk) and relation to *non-ultralocality* (cf. Magro's talk)

- Role of secret symmetry, quantum double, universal R-matrix ? Relation to Q-operator [Bazhanov,Frassek,Łukowski,Meneghelli,Staudacher]

- Exciting links to condensed matter theory (Hubbard model) and Pohlmeyer reduction [Grigoriev, Tseytlin, Hoare, Hollowood, Miramontes]

  (cf. Hoare's and Van Tongeren's talks)

- Fascinating connections with Yangian and dual superconformal symmetries of scattering amplitudes await to be fully investigated

- Hopf algebra of symbols (cf. Duhr's talk). Any connection?

THANK YOU!