Harmonic R-matrices for the $\mathcal{N} = 4$ amplitudes

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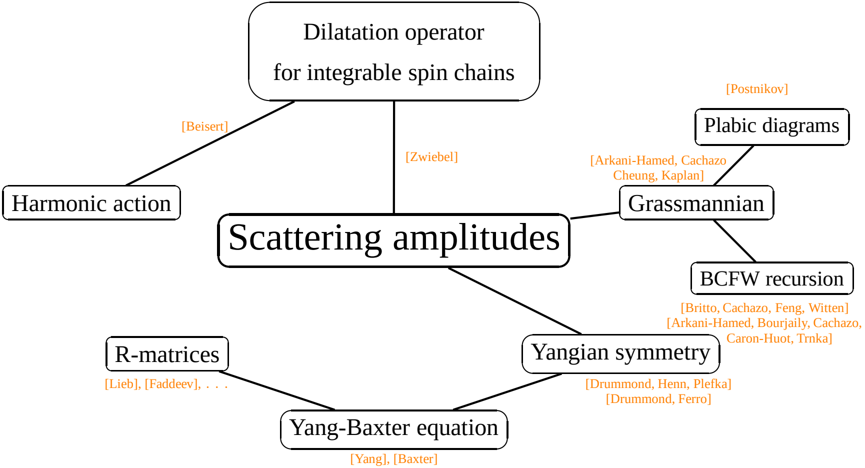
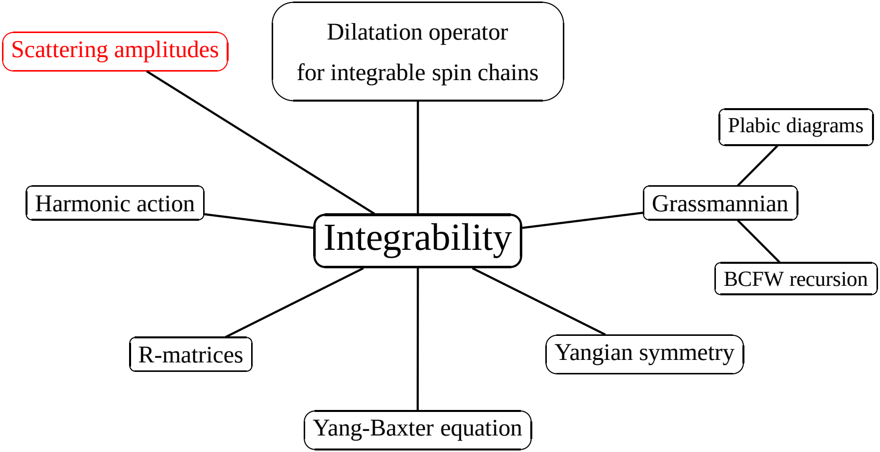
Outline



Dilatation operator
for integrable spin chains
[Beisert]
[Zwiebel]
Harmonic action
Scattering amplitudes
[Postnikov]
Plabic diagrams
[Arkani-Hamed, Cachazo
Cheung, Kaplan]
Grassmannian
BCFW recursion
[Britto, Cachazo, Feng, Witten]
[Arkani-Hamed, Bourjaily, Cachazo,
Caron-Huot, Trnka]
R-matrices
[Lieb], [Faddeev], . . .
Yangian symmetry
[Drummond, Henn, Plefka]
[Drummond, Ferro]
Yang-Baxter equation
[Yang], [Baxter]

Outline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Amplitudes

We consider color-ordered scattering amplitudes of superfields

$$\Phi = G^{+} + \eta^{A}\Gamma_{A} + \frac{1}{2!}\eta^{A}\eta^{B}S_{AB} + \frac{1}{3!}\eta^{A}\eta^{B}\eta^{C}\epsilon_{ABCD}\overline{\Gamma}^{D} + \frac{1}{4!}\eta^{A}\eta^{B}\eta^{C}\eta^{D}\epsilon_{ABCD}G^{-}$$

The amplitudes $\mathcal{A}_{n,k}$ are labeled by two numbers:

- number of particles – n
- helicity – η^{4k}

All particles are massless: $p^2 = 0 \Rightarrow p^{\alpha\dot{\alpha}} = \lambda^{\alpha}\tilde{\lambda}^{\dot{\alpha}}$

On-shell superspace – $\Lambda^{\mathcal{A}} = (\lambda^{\alpha}, \tilde{\lambda}^{\dot{\alpha}}, \eta^{A})$

The simplest non-trivial examples are MHV amplitudes – Parke-Taylor formula for the tree-level: [Parke, Taylor]

$$\mathcal{A}_{n,2} = \frac{\delta^4(P)\delta^8(Q)}{\langle 12\rangle\langle 23\rangle\ldots\langle n1\rangle}, \qquad Q^{\alpha A} = \sum_i \lambda_i^{\alpha}\eta_i^{A}, \qquad P^{\alpha\dot{\alpha}} = \sum_i \lambda_i^{\alpha}\tilde{\lambda}_i^{\dot{\alpha}}$$

Symmetries of amplitudes

At tree level:

- Superconformal symmetry (ignoring collinear anomalies) [Witten]

$$j_a \mathcal{A}_n = 0\,, \qquad j_a \in \mathfrak{psu}(2,2|4)$$

- Dual superconformal symmetry - superconformal symmetry in dual space [Drummond, Henn, Korchemsky, Sokatchev]

$$x_i^{\alpha\dot{\alpha}} - x_{i+1}^{\alpha\dot{\alpha}} = \lambda_i^\alpha \tilde{\lambda}_i^{\dot{\alpha}}\,, \qquad \theta_i^{\alpha A} - \theta_{i+1}^{\alpha A} = \lambda_i^\alpha \eta_i^A$$

  Then

$$j'_a \mathcal{A}_n = 0\,, \qquad j'_a \in \mathfrak{psu}(2,2|4)^{\text{Dual}}$$

- Superconformal + dual superconformal symmetry $\longrightarrow$ **Yangian symmetry** [Drummond, Henn, Plefka]

Hint of integrability

Anomalies for loop amplitudes arise from infrared divergences

Yangian $Y(g)$

$$\left.\begin{array}{c} \text{level-zero generators } j_a : [j_a, j_b] = f_{ab}^c j_c \\ \text{level-one generators } j_a^{(1)} : [j_a, j_b^{(1)}] = f_{ab}^c j_c^{(1)} \\ \text{Serre relations} \end{array}\right\} \Longrightarrow \text{Infinite dimensional algebra } Y(g)$$

Yangian invariance of the $\mathcal{N} = 4$ tree-level amplitudes

$$y\mathcal{A}_n = 0\,, \qquad y \in Y(\mathfrak{psu}(2,2|4))$$

Twistor representation $\mathcal{Z}^{\mathcal{A}} = (\tilde{\mu}^{\alpha}, \tilde{\lambda}^{\dot{\alpha}}, \eta^A)$

$$j_{\mathcal{B}}^{\mathcal{A}} = \sum_i \mathcal{Z}_i^{\mathcal{A}} \frac{\partial}{\partial \mathcal{Z}_i^{\mathcal{B}}}$$

$$(j^{(1)})_{\mathcal{B}}^{\mathcal{A}} = \sum_{i<j} (-1)^{\mathcal{C}} \left[\mathcal{Z}_i^{\mathcal{A}} \frac{\partial}{\partial \mathcal{Z}_i^{\mathcal{C}}} \mathcal{Z}_j^{\mathcal{C}} \frac{\partial}{\partial \mathcal{Z}_j^{\mathcal{B}}} - (i \leftrightarrow j) \right]$$

For more details see Song He's talk

- What about other realizations of the Yangian? Alessandro Torrielli advertised Drinfeld's second realization, but we focus on the RTT definition
- Another approach to the Yangian algebra: monodromy matrices

$$\mathcal{M}(z) = R_1(z) \dots R_L(z)$$

are generating functions for all levels of the Yangian generators

$$\mathcal{M}(z)^{\mathcal{A}}_{\mathcal{B}} = \sum_{i=0}^{\infty} \frac{1}{z^i} (j^{(i)})^{\mathcal{A}}_{\mathcal{B}}$$

- In order to find R-matrices $R(z)$ we have to solve the **Yang-Baxter equation**
- Transfer matrices proved their usefulness in the theory of integrable models (Quantum Inverse Scattering Method)

What is the relation between these two approaches?

How amplitudes fit into the well-known theory of integrable models?

What is the role of Yang-Baxter equation?

What about the spectral parameter?

- Zwiebel observed the relation between amplitudes and the dilatation operator for the integrable spin chain [Zwiebel]

$$\langle \Lambda_1 \Lambda_2 | \mathcal{D}_{L\to 2} | \Lambda_3 \dots \Lambda_{L+2} \rangle = \mathcal{A}_{L+2}(\Lambda_1, \dots, \Lambda_{L+2})$$

- Particularly interesting for $L = 2$ because $\mathcal{D}_{2\to 2} = \mathcal{H}$, where $\mathcal{H}$ is the nearest-neighbor Hamiltonian for $\mathcal{N} = 4$ SYM integrable spin chain

$$\begin{array}{ccc} \text{Spin chain} & & \text{Amplitudes} \\ \mathcal{H} = \mathcal{D}_{2\to 2} & \overset{Zwiebel}{\longleftrightarrow} & \mathcal{A}_4 \\ \uparrow d\log & & \uparrow \\ R(z) & \longleftrightarrow & ? \end{array}$$

- Simple form of the Hamiltonian in terms of the harmonic action

Harmonic action

- Oscillator representation of the spin chain states

$$|\text{two-site state}\rangle = \prod_{i=1}^{n} \mathbf{A}_{p_i}^{\dagger\, A_i}|00\rangle\,, \qquad p_i \in \{1,2\}$$

  where

$$\left[\mathbf{A}_{p_i,A_i}, \mathbf{A}_{p_j}^{\dagger A_j}\right] = \delta_{A_i}^{A_j}\,\delta_{p_i,p_j}$$

- Harmonic action of the Hamiltonian density [Beisert]

$$\mathcal{H}_{12}|\text{two-site state}\rangle = \sum c_{n,n_{12},n_{21}}\delta_{C_1,0}\delta_{C_2,0}|\text{ another two-site state}\rangle$$

  where

$$\begin{aligned} c_{n,n_{12},n_{21}} &= (-1)^{1+n_{12}n_{21}}\frac{\Gamma(\frac{n_{12}+n_{21}}{2})\Gamma(1+\frac{n-n_{12}-n_{21}}{2})}{\Gamma(1+\frac{n}{2})} \\ c_{n,0,0} &= h\left(\frac{n}{2}\right) \end{aligned}$$

- It gives the same results as

$$\mathcal{H} = 2(\psi(\mathbb{J}+1)-\psi(1))\,, \qquad \mathbb{J}(\mathbb{J}+1) = C_2$$

Basis of hops

- It is an integrable Hamiltonian as it is generated by an R-matrix

$$\mathcal{H} = \frac{d}{dz} \log R(z) \Big|_{z=0}$$

where

$$R(z) = (-1)^{\mathbb{J}} \frac{\Gamma(z + \mathbb{J} + 1)}{\Gamma(-z + \mathbb{J} + 1)}, \qquad \mathbb{J}(\mathbb{J} + 1) = C_2$$

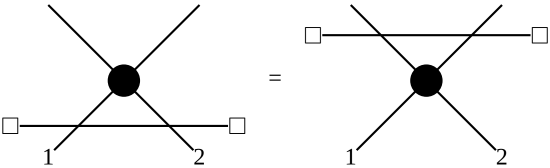
- Difficult to directly see the relation to the harmonic action. Can we express this R-matrix in the harmonic action language - harmonic R-matrix?
- Focus on compact representations (one-row) and introduce

$$\mathbf{Hop}_i = \frac{1}{i!i!} \mathbf{A}_1^{\dagger A_1} \mathbf{A}_1^{\dagger A_2} \ldots \mathbf{A}_1^{\dagger A_i} \mathbf{A}_2^{\dagger B_1} \ldots \mathbf{A}_2^{\dagger B_i} \mathbf{A}_{1,B_1} \ldots \mathbf{A}_{1,B_i} \mathbf{A}_{2,A_1} \ldots \mathbf{A}_{2,A_i}$$

Then

$$\mathcal{H}_{12} = \sum_{i=0}^{s} c_{2s,i,i} \, \mathbf{Hop}_i$$

Solve the Yang-Baxter equation



$$R(z_1-z_2)\left(z_1\delta_A^B+(J_1)_A^B\right)\left(z_2\delta_B^C+(J_2)_B^C\right)=\left(z_2\delta_A^B+(J_2)_A^B\right)\left(z_1\delta_B^C+(J_1)_B^C\right)R(z_1-z_2)$$

where $(J_i)_B^A=\mathbf{A}_i^{\dagger A}\mathbf{A}_{i,B}$. Using the following ansatz

$$R(z)=\sum_i \alpha_i(z)\mathbf{Hop}_i$$

one finds

$$R(z)=\underbrace{\frac{\Gamma(z+1)\Gamma(z+1)\Gamma(-z+s+1)}{\Gamma(z+s+1)}}_{\rho(z)}\sum_{i=0}^{s}\frac{\sin\pi(-z+s-i)}{\pi}B(-z+s-i,i+1)\mathbf{Hop}_i$$

The right hand side is finite for all i – no need for regularization

Harmonic R-matrix (2)

- Taking the logarithmic derivative of the harmonic R-matrix one finds the Hamiltonian in the harmonic action form
- Why is it relevant to amplitudes?

  One can rewrite

$$\begin{aligned}\langle \mathbf{A}_1 \mathbf{A}_2 | R(z) | \mathbf{A}_1' \mathbf{A}_2' \rangle &= \int dc_{11} dc_{12} dc_{21} dc_{22} \, f(c_{11}, c_{12}, c_{21}, c_{22}; z) \times \\ & \times \delta(\mathbf{A}_1' + c_{11}\mathbf{A}_1 + c_{12}\mathbf{A}_2) \delta(\mathbf{A}_2' + c_{21}\mathbf{A}_1 + c_{22}\mathbf{A}_2)\end{aligned}$$

- Similar to the Grassmannian formula for amplitudes

Amplitudes in the Grassmannian language

[Arkani-Hamed, Bourjaily, Cachazo, Caron-Huot, Cheung, Goncharov, Hodges, Kaplan, Postnikov, Trnka]

$$\mathcal{L}_{n,k} = \int \frac{\prod_{a=1}^{k} \prod_{i=1}^{n} dc_{ai}}{\mathcal{M}_1 \mathcal{M}_2 \ldots \mathcal{M}_n} \prod_{a=1}^{k} \delta^{4|4}\left(\sum_{i=1}^{n} c_{ai} \mathcal{Z}_i \right)$$

- $\mathcal{Z}^{\mathcal{A}} = (\tilde{\mu}^{\alpha}, \tilde{\lambda}^{\dot{\alpha}}, \eta^{A})$
- $C = (c_{ai})$ - $k \times n$ complex matrix
- $\mathcal{M}_p = (p \ldots p + k - 1)$ - determinants of $k \times k$ submatrices
- gauge freedom - $GL(k)$

Example

$$\mathcal{L}_{4,2} = \int \frac{dc_{13} dc_{14} dc_{23} dc_{24}}{c_{13} c_{24} (c_{13} c_{24} - c_{14} c_{23})} \delta^{4|4}(\mathcal{Z}_1 + c_{13}\mathcal{Z}_3 + c_{14}\mathcal{Z}_4) \delta^{4|4}(\mathcal{Z}_2 + c_{23}\mathcal{Z}_3 + c_{24}\mathcal{Z}_4)$$

$$C = \begin{pmatrix} 1 & 0 & c_{13} & c_{14} \\ 0 & 1 & c_{23} & c_{24} \end{pmatrix}$$

Uniqueness of Yangian invariants

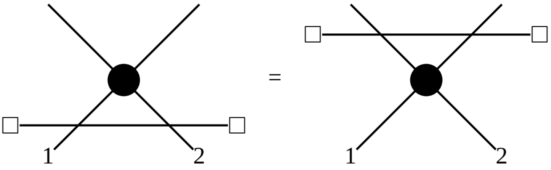
- The formula for $\mathcal{L}_{n,k}$ is Yangian invariant (up to total derivatives)
- It is uniquely determined by the requirements of Yangian symmetry, cyclicity, and physical helicities [Drummond, Ferro], [Korchemsky, Sokatchev]
- If one relaxes the helicity condition more general Yangian invariants can be written

$$Inv_{n,k} = \int \frac{\prod_{a=1}^{k}\prod_{i=1}^{n} dc_{ai}}{\mathcal{M}_1\mathcal{M}_2\ldots\mathcal{M}_n} F(C) \prod_{a=1}^{k} \delta^{4|4}\left(\sum_{i=1}^{n} c_{ai}\mathcal{Z}_i\right)$$

where $F(C)$ satisfies some differential equation

YBE in the Grassmannian language

Yang-Baxter equation:



$$R(z)\left(z\,(J_2)_A^C + (-1)^B (J_1)_A^B\,(J_2)_B^C\right) = \left(z\,(J_2)_A^C + (-1)^B (J_1)_A^B\,(J_2)_B^C\right) R(z)$$

with ansatz

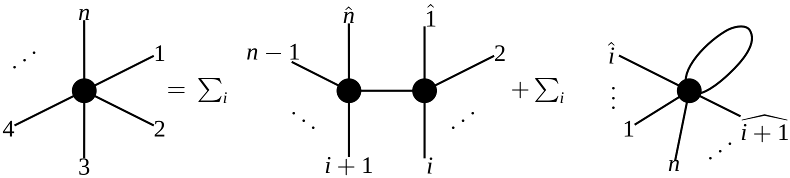
$$\mathcal{R}(z) = \int \frac{dc_{13}dc_{14}dc_{23}dc_{24}}{c_{13}c_{24}(c_{13}c_{24} - c_{14}c_{23})} F(C;z)\,\delta^{4|4}(\mathcal{Z}_1 + c_{13}\mathcal{Z}_3 + c_{14}\mathcal{Z}_4)\,\delta^{4|4}(\mathcal{Z}_2 + c_{23}\mathcal{Z}_3 + c_{24}\mathcal{Z}_4)$$

and the fact that all particles have physical helicities leads to the answer

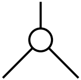
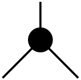
$$F(C;z) = \left(\frac{c_{13}c_{24}}{c_{13}c_{24} - c_{14}c_{23}}\right)^z$$

BCFW

- What about general scattering?
- All-loop recursion relation [Arkani-Hamed, Bourjaily, Cachazo, Caron-Huot, Trnka]



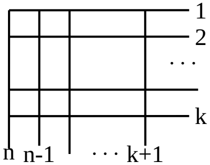
- Using BCFW recursion relations we can write all amplitudes employing only two objects:

$\mathcal{A}_{3,1} =$  $\mathcal{A}_{3,2} =$ 

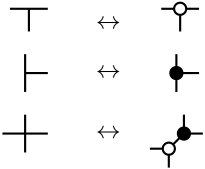
- Amazingly, one can express all amplitude using only on-shell diagrams

Amplitudes from on-shell diagrams

For tree level:

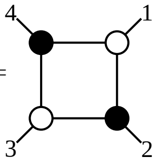


Dictionary [Postnikov]

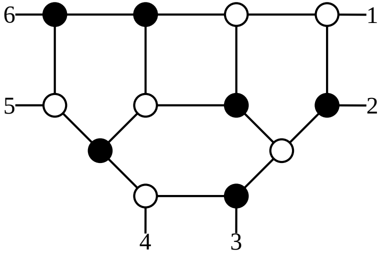


Examples:

$\mathcal{A}_{4,2} =$



$\mathcal{A}_{6,3} =$



These are tree-level diagrams even though we see loops here!

- Can we find objects which after gluing into on-shell diagram produce R(z)?
- What kind of equations do they satisfy?
- Again there are two objects: MHV R-matrix – $R_\bullet(z_1, z_2)$ and $\overline{\text{MHV}}$ R-matrix – $R_\circ(z_1, z_2)$

- Similar to the bootstrap equation (see Torrielli's talk)
- They satisfy the following equations:

$$
\begin{aligned}
z_1 (J_1)^A_C \mathcal{R}_\bullet(z_1, z_2) &= \mathcal{R}_\bullet(z_1, z_2)(J_1)^A_B(z_1 \delta^B_C + (J_2)^B_C) \\
(J_1)^A_B(z_1 \delta^B_C + (J_2)^B_C)\mathcal{R}_\circ(z_1, z_2) &= z_1 \mathcal{R}_\circ(z_1, z_2)(J_1)^A_C
\end{aligned}
$$

- A second set of equations with $(1 \leftrightarrow 2)$ leads to a second spectral parameter z_2

Interpretation of the spectral parameters

Solutions

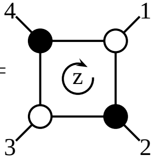
$$
\begin{aligned}
\mathcal{R}_\bullet(z_1,z_2) &= \oint \frac{dc_1 dc_2}{c_1 c_2}\frac{1}{c_1^{z_1}c_2^{z_2}}\delta^{4|4}(\mathcal{Z}_1^{\mathcal{A}}+c_1\mathcal{Z}_3^{\mathcal{A}})\delta^{4|4}(\mathcal{Z}_2^{\mathcal{A}}+c_2\mathcal{Z}_3^{\mathcal{A}})\\
\mathcal{R}_\circ(z_1,z_2) &= \oint \frac{dc_1 dc_2}{c_1 c_2}\frac{1}{c_1^{z_1}c_2^{z_2}}\delta^{4|4}(\mathcal{Z}_3^{\mathcal{A}}+c_1\mathcal{Z}_1^{\mathcal{A}}+c_2\mathcal{Z}_2^{\mathcal{A}})
\end{aligned}
$$

- Why 2 spectral parameters?
- We know that 3-point amplitudes are very singular objects – if we demand all legs to be on-shell, with vanishing central charges and real momenta then 3-point amplitudes vanish. We have to complexify momenta to find the non-trivial answer
- If we want to find non-trivial R-matrices we have to in addition relax the central charge (helicity) constraints

$$
\begin{aligned}
(J_1)_A^A\mathcal{R}_\circ(z_1,z_2) &= z_1\,\mathcal{R}_\circ(z_1,z_2)\\
(J_2)_A^A\mathcal{R}_\circ(z_1,z_2) &= z_2\,\mathcal{R}_\circ(z_1,z_2)\\
\mathcal{R}_\circ(z_1,z_2)(J_3)_A^A &= (z_1+z_2)\mathcal{R}_\circ(z_1,z_2)
\end{aligned}
$$

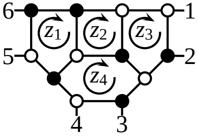
The spectral parameters have the interpretation of unphysical particle helicities!

- For 4 points we get one non-vanishing spectral parameter after demanding that outer particles have vanishing central charge

$\mathcal{R}_{4,2} =$ 

$$\oint \frac{dc_{13}dc_{14}dc_{23}dc_{24}}{c_{13}c_{24}(c_{13}c_{24}-c_{14}c_{23})} \left(\frac{c_{13}c_{24}}{c_{13}c_{24}-c_{14}c_{23}}\right)^{z} \delta^{4|4}(C\cdot\mathcal{Z})$$

- For higher-point harmonic R-matrices the number of parameters grows and is equal to the number of loops in the on-shell diagram

$\mathcal{R}_{6,3} =$ 

$$\oint \frac{d^9C}{(123)(234)\dots(612)}\delta^{4|4}(C\cdot\mathcal{Z}) \left(\frac{c_{36}(c_{16}c_{25}-c_{15}c_{26})}{c_{16}(c_{26}c_{35}-c_{25}c_{36})}\right)^{z_1} \left(\frac{c_{15}c_{26}\det C}{c_{16}(c_{14}c_{25}-c_{15}c_{24})(c_{25}c_{36}-c_{26}c_{35})}\right)^{z_2} \left(\frac{c_{14}(c_{15}c_{26}-c_{16}c_{25})}{c_{16}(c_{14}c_{25}-c_{15}c_{24})}\right)^{z_3} \left(\frac{c_{15}c_{26}}{(c_{16}c_{25}-c_{15}c_{26})}\right)^{z_4}$$

Loops

- One can write any loop amplitude using on-shell diagrams

$$\mathcal{A}_{n,k}^{\ell} \longrightarrow \mathcal{A}_{n-2,k-1}^{\ell+1}$$

- The simplest example is $\mathcal{A}_{6,3}^{\text{tree-level}} \longrightarrow \mathcal{A}_{4,2}^{\text{1-loop}}$. One gets:

$$\mathcal{A}_{4,2}^{\text{1-loop}} = \mathcal{A}_{4,2}^{\text{tree-level}} \int \frac{d^4 q}{q^2 (q+p_1)^2 (q+p_1+p_2)^2 (q+p_1+p_2+p_3)^2}$$

- Tree-level amplitudes factorize, and we are left with the box integral (B)
- The box integral is known to be divergent $\rightarrow$ one uses dimensional or mass regularization to calculate it

$$B \sim \frac{2}{\epsilon^2}\left(\left(\frac{s}{\mu^2}\right)^{-\epsilon} + \left(\frac{t}{\mu^2}\right)^{-\epsilon}\right) - \log^2\left(\frac{s}{t}\right) - \frac{4\pi^2}{3}$$

Exciting possibility: spectral regularization

- Use the spectral parameter(s) to regulate loop amplitudes in a novel way
- Should provide a symmetry-preserving regularization scheme
- In particular, we stay in exactly four dimensions: replace *dimensional* regularization by *spectral* regularization
- Currently under active investigation

Summary and outlook

- We constructed a new class of objects which are deformations of the $\mathcal{N} = 4$ amplitudes
- We introduced spectral parameters into the scattering amplitude problem

What is it good for?

- Analyticity requirements in the spectral parameter plane should fix contours of integrations
- Possibility to provide a novel symmetry preserving regulator
- Should allow to establish the exact link between the amplitude and the spectral problem

Outlook:

- Introduction of the spectral parameter was the key to solve the AdS/CFT all-loop spectral problem (ABA, TBA, Y-system). Hopefully history will repeat itself !