Y-system and Bethe Ansatz
Bethe Ansatz
Y-system
Hirota equation

Reduction of the Y-system to a FiNLIE
Q-functions
Analyticity
FiNLIE

Weak coupling behaviour of the FiNLIE
Iterative structure
How ζ functions appear
6-loop Konishi energy

Using FiNLIE for AdS/CFT spectrum.

Sébastien Leurent
LPT-ENS (Paris)

based on [arXiv:1110.0562] (N. Gromov, V. Kazakov, SL & D. Volin)
and a collaboration with D. Serban and D. Volin.

ETH-Zürich, 23 August 2012

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   - The Y-system for AdS/CFT
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   - Q-functions and Wronskian parameterization of the Y-system
   - Additional analyticity properties
   - Structure of the FiNLIE
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Bethe Ansatz

As explained in previous talks, the Bethe Ansatz gives

- the eigenstates of the dilation operator
- the eigenvalues (ie the dimensions of operator)

as soon as we solve the Bethe Equation

$$\forall k, \quad e^{iLp_k} = \prod_{j\neq k} S_{jk}$$

AdS/CFT case

For AdS_5/CFT_4, this ansatz only describes the "long" operators. This talk will focus on the dimensions of short operator.

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The Asymptotic Bethe Ansatz gives

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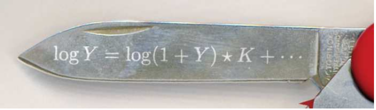
The Y-system for AdS/CFT

Spectrum of excited states in AdS/CFT

Each excited state is associated to an infinite set of Y-functions, which solve the

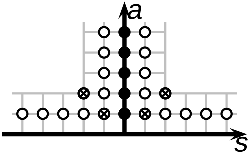
TBA equations



$$\log Y_{a,s} = \sum \log(1+Y_{a',s'})\star K_{a',s',a,s}$$

[Bombardelli Fioravanti Tateo 09][Gromov Kazakov Kozak Vieira 09][Arutyunov Frolov 09]

Y-system



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[Gromov Kazakov Vieira 09]

$$f^\pm \equiv f(u \pm \tfrac{i}{2})$$

Finite-size spectrum of AdS/CFT

Energy of a state : $E = -\sum_{a,s} \int E_{a,s}(u)\ \log\left(1 + Y_{a,s}(u)\right)\ \mathrm{d}u$

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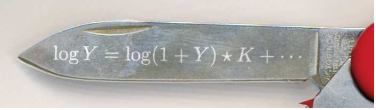

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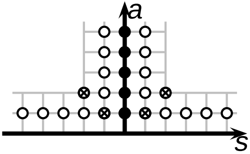

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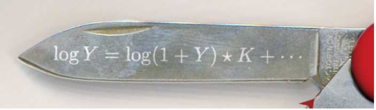

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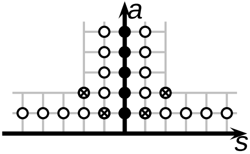

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- These formulations are equivalent up to analyticity conditions clarified in [Cavaglia Fioravanti Tateo 09]

  Branch points originally come from $x = \frac{u}{2g} + i\sqrt{1 - \frac{u^2}{4g^2}}$

Y-system ↔ Hirota equation

(also called T-system)

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$$T_{a,s}^{+}T_{a,s}^{-}=T_{a+1,s}T_{a-1,s}+T_{a,s+1}T_{a,s-1}$$

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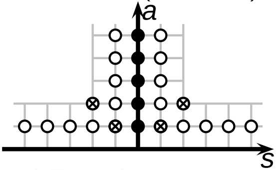

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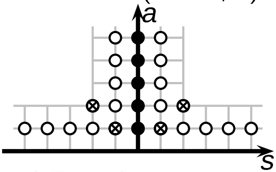

Y-system ↔ Hirota equation

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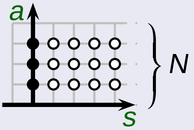
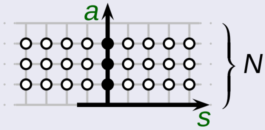
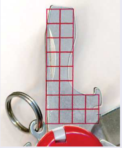
Other Y and T-systems

$SU(N)$ Gross-Neveu	$SU(N)\times SU(N)$ Principal Chiral Model	$SU(2\|2)$ Spin Chain
		

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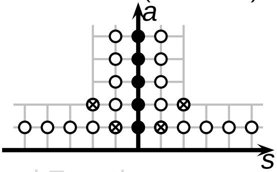

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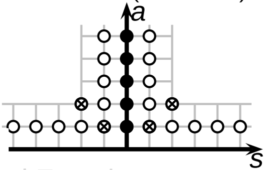

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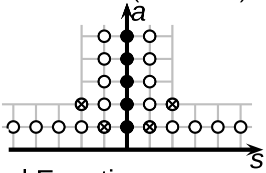

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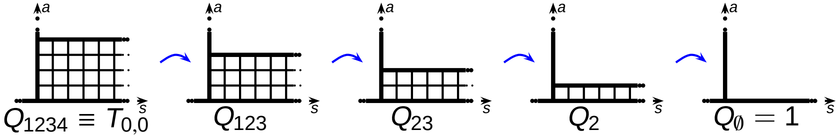
Q-functions and Hasse diagram

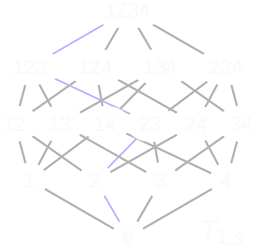
example of the SU(4) (a,s)-lattice

- "Undressing" procedure (Bäcklund Transformation)

su(4) ⊃ su(3) ⊃ su(2) ⊃ su(1)



- N! different "nesting paths" define 2^N Q-functions



- Related by QQ-relations :

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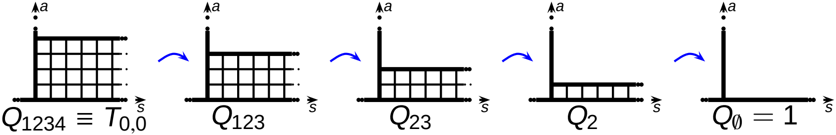

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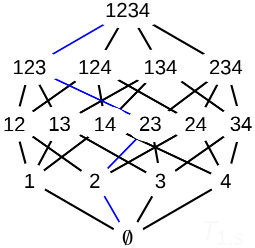

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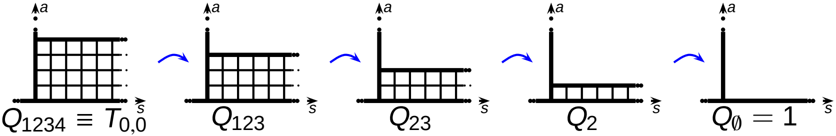

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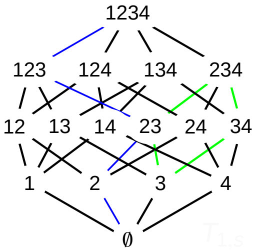

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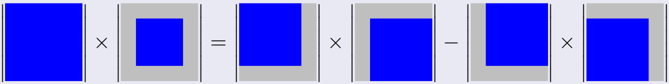
Jacobi-Trudi identity : for an arbitrary determinant



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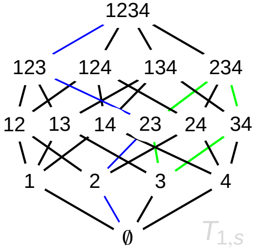

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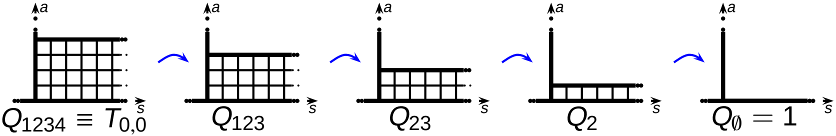

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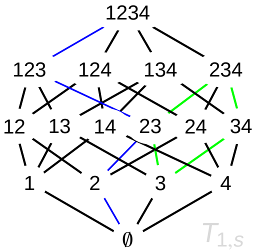

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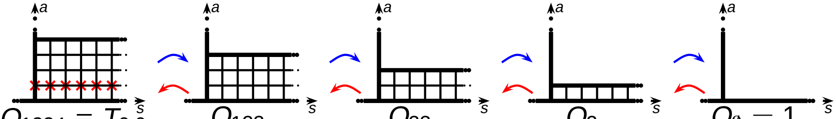

$Q_{1234} \equiv T_{0,0}$ $\quad Q_{123}$ $\quad Q_{23}$ $\quad Q_{2}$ $\quad Q_{\emptyset} = 1$

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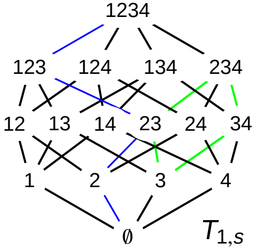

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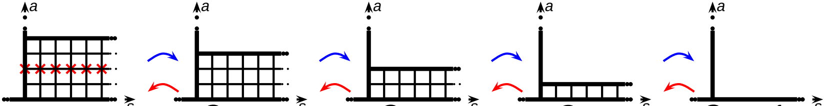
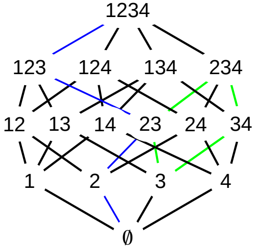


$Q_{1234} \equiv T_{0,0}$ $\quad Q_{123}$ $\quad Q_{23}$ $\quad Q_2$ $\quad Q_\emptyset = 1$

- N! different "nesting paths" define 2^N Q-functions



- Related by QQ-relations :

$$Q_{234} Q_3 = \begin{vmatrix} Q_{23}^{+} & Q_{34}^{+} \\ Q_{23}^{-} & Q_{34}^{-} \end{vmatrix}$$

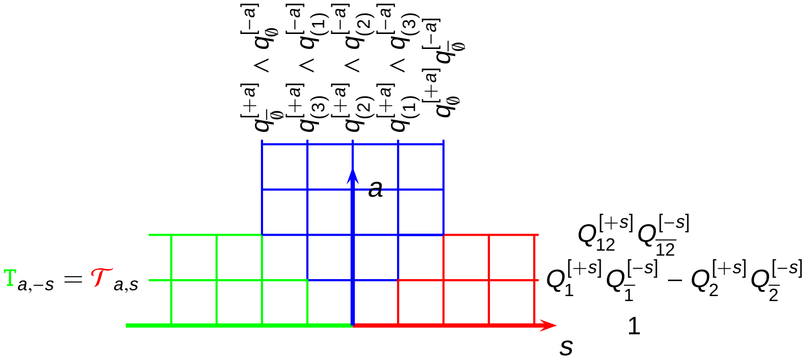
- Wronskian solution :

$$Q_{\bar{4}} \equiv Q_{234} = \begin{vmatrix} Q_2^{++} & Q_3^{++} & Q_4^{++} \\ Q_2 & Q_3 & Q_4 \\ Q_2^{--} & Q_3^{--} & Q_4^{--} \end{vmatrix}$$

- giving all T-functions :

$$T_{2,s} = Q_{12}^{[+s]} Q_{\overline{12}}^{[-s]} - Q_{13}^{[+s]} Q_{\overline{13}}^{[-s]} + \cdots$$

Wronskian solution of Hirota equation for AdS/CFT.



where

$$q_{(1)}^{[+a]} \wedge q_{(3)}^{[-a]} \equiv q_1^{[+a]} q_{\bar 1}^{[-a]} - q_2^{[+a]} q_{\bar 2}^{[-a]} + q_3^{[+a]} q_{\bar 3}^{[-a]} - q_4^{[+a]} q_{\bar 4}^{[-a]}.$$

⇝ **FiNLIE** [Gromov Kazakov S.L. Volin 11]

Parameterizes the Y-system into a finite set of functions.
A Finite set of Non-Linear Integral Equations can be derived.

Wronskian solution of Hirota equation ⇝ FiNLIE.

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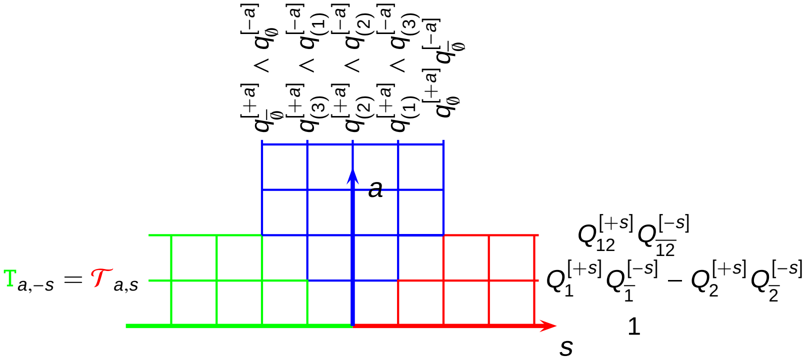

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In the classical limit, $T_{a,s}(u) = \chi_{a,s}\left(\Omega(u)\right)$ where $\Omega \in U(2,2|4)$.

characters in rectangular irreps [Gromov Kazakov Tsuboi 10]

- Actually, $\Omega \in PSU(2,2|4) \Rightarrow$ more constraints :
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  ⇝ extra symmetries of the characters.
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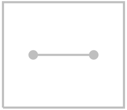

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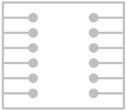

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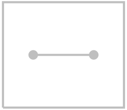

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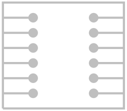

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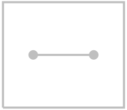

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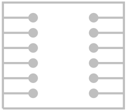

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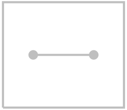

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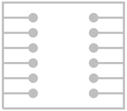

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Using FiNLIE for AdS/CFT spectrum.

S. Leurent

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Bethe Ansatz
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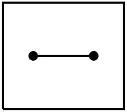

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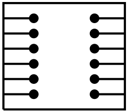

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Structure of the FiNLIE

[Gromov Kazakov S.L. Volin 11]

- Y-stem parameterized by three functions
  - two real densities ρ and ρ_2 with finite support $[-2g, 2g]$.
  - one gauge function U (reduces to a real function on the real axis)

they define 3 Q-functions

$\rightsquigarrow$ other functions obtained by QQ-relations

More details

- + *Exact Bethe Equation* for the zeroes of T-functions

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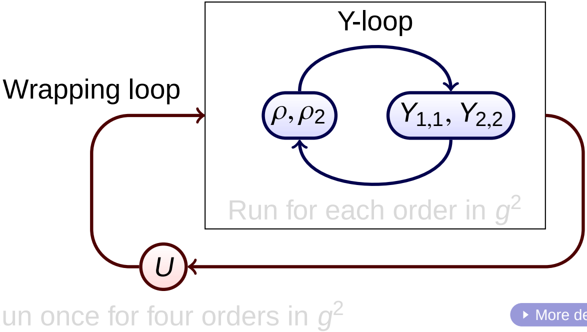

$$g^2 = \frac{\lambda}{(4\pi)^2}$$

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Outline

1. Y-system and Bethe Ansatz
   - Bethe Ansatz
   - The Y-system for AdS/CFT
   - Hirota equation
2. Reduction of the Y-system to a FiNLIE
   - Q-functions and Wronskian parameterization of the Y-system
   - Additional analyticity properties
   - Structure of the FiNLIE
3. Weak coupling behaviour of the FiNLIE
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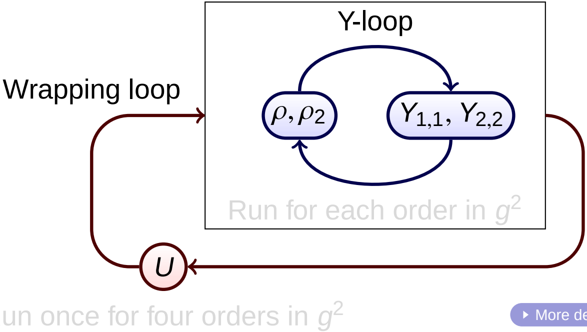

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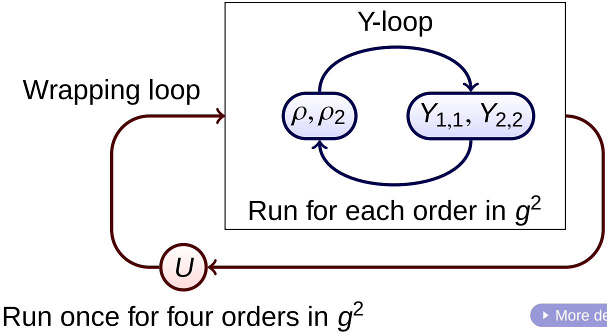

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QQ-relations and pole structure

- Parameterization $\rightsquigarrow$ QQ-relations :

$$q_1 = 1, \quad q_{12} \equiv \tilde{Q}, \quad q_{13}, \quad q_{123} \equiv U$$

$$q_1\, q_{123} = \begin{vmatrix} q_{12}^+ & q_{13}^+ \\ q_{12}^- & q_{13}^- \end{vmatrix} \Rightarrow \left(\frac{q_{13}}{q_{12}}\right)^- - \left(\frac{q_{13}}{q_{12}}\right)^+ = \frac{q_1\, q_{123}}{q_{12}^+ q_{12}^-}$$

$$q_{13} = q_{12} \sum_{k=0}^{\infty} \left(\frac{q_1\, q_{123}}{q_{12}^+ q_{12}^-}\right)^{[2k+1]}$$

- Leading order : $U \simeq -2\frac{g^4}{u^2}$, $q_{1,2} \simeq Q \equiv (u-u_1)(u+u_1)$.

$$\Rightarrow q_{13} \simeq Q \sum_{k=0}^{\infty} \left(\frac{U}{Q^+Q^-}\right)^{[2k+1]}$$

$$f^\pm \equiv f(u \pm \tfrac{i}{2})$$

- Bethe equations make the poles from $\frac{1}{Q^+Q^-}$ cancel

$$\rightsquigarrow q_{13} \simeq 18 g^4 \left(-i\,u + Q\,\psi(-i\,u + \tfrac{1}{2})\right)$$

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Strategy of weak coupling expension

- FiNIE involves integrals of functions which (like q_{13}) have inifinitely many branch points.
- These functions become Multiple-Zeta-Functions, ie $\sum_{k_1,k_2,\ldots\in\mathbb{N}}\left(\frac{1}{(u+i\,k_1)^{n_1}}\frac{1}{(u+i\,k_1+i\,k_2)^{n_2}}\cdots\right)$ at weak coupling

where branch points originally come from $x=\frac{u}{2g}+\sqrt{\frac{u^2}{4g^2}-1}$

- Integrals are computed by closing the contour at infinity The sum of residues gives some $\zeta(n)$

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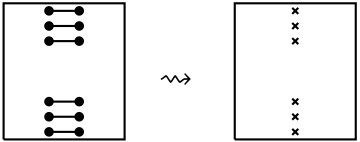

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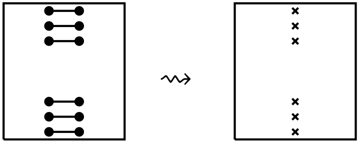

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Result

Energy of the Konishi state at 6-loops

- Asymptotic Bethe ansatz gives

$$
\begin{aligned}
E_{ABA} =& 2 + 12g^2 - 48g^4 + 336\,g^6 \\
& - (2820 + 288\,\zeta_3)g^8 + (26508 + 4320\,\zeta_3 + 2880\,\zeta_5)\,g^{10} \\
& - (269148 + 55296\,\zeta_3 + 44064\,\zeta_5 + 30240\,\zeta_7)\,g^{12}\,.
\end{aligned}
$$

- From FiNLIE, we analytically derive the correction

$$
\begin{aligned}
E - E_{ABA} =& (324 + 864\,\zeta_3 - 1440\,\zeta_5)g^8 \\
& + (-11340 + 2592\,\zeta_3 - 11520\,\zeta_5 - 5184\,\zeta_3^2 \\
& + 30240\,\zeta_7)g^{10}
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[Bajnok Egedüs Janik Łukowski 09]

[Eden Heslop Korchemsky Smirnov Sokatchev 12]

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\begin{aligned}
& + (261468 - 207360\zeta_3 - 20736\zeta_3^2 + 156384\zeta_5 \\
& + 155520\zeta_3\zeta_5 + 105840\zeta_7 - 489888\zeta_9)g^{12}\,.
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Discrepency

There is a mismatch with the (upcoming) result of [Bajnok Janik 12], which gives

$$(261468 - 215136\zeta_3 - 41472\zeta_3^2 + 156384\zeta_5 + 190080\zeta_3\zeta_5 + 105840\zeta_7 - 489888\zeta_9)g^{12}$$

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Conclusion

- FiNLIE rewrites the Y-system
  - with new symmetries identified
  - as a Finite set of NLIEs
- It can be used to explicitly solve the Y-system orders by orders in perturbation theory
  - 6-loops result to be confirmed / infirmed.
  - 7-loops is not conceptually more complicated
  - double wrapping is in principle accessible too
- giving a better understanding of some analytical properties
  - Exact Bethe equation $\leftrightarrow$ absence of poles
- Raising open questions
  - Spin chain interpretation of FiNLIE at weak coupling ?
  - Strong coupling expension
  - FiNLIE from first principles

Y-system and Bethe Ansatz
Bethe Ansatz
Y-system
Hirota equation
Reduction of the Y-system to a FiNLIE
Q-functions
Analyticity
FiNLIE
Weak coupling behaviour of the FiNLIE
Iterative structure
How ζ functions appear
6-loop Konishi energy

Conclusion

- FiNLIE rewrites the Y-system
  - with new symmetries identified
  - as a Finite set of NLIEs
- It can be used to explicitly solve the Y-system orders by orders in perturbation theory
  - 6-loops result to be confirmed / infirmed.
  - 7-loops is not conceptually more complicated
  - double wrapping is in principle accessible too
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finally

Thank you !

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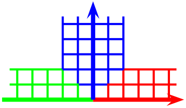
AdS/CFT FiNLIE

$$
\begin{aligned}
T_{a,+1} &= q_1^{[+a]}\bar{q}_2^{[-a]} + q_2^{[+a]}\bar{q}_1^{[-a]} + q_3^{[+a]}\bar{q}_4^{[-a]} + q_4^{[+a]}\bar{q}_3^{[-a]} \,, \\
T_{a,0} &= q_{12}^{[+a]}\bar{q}_{12}^{[-a]} + q_{34}^{[+a]}\bar{q}_{34}^{[-a]} - q_{14}^{[+a]}\bar{q}_{14}^{[-a]} \\
&\quad - q_{23}^{[+a]}\bar{q}_{23}^{[-a]} - q_{13}^{[+a]}\bar{q}_{24}^{[-a]} - q_{24}^{[+a]}\bar{q}_{13}^{[-a]} \,, \\
q_\emptyset q_{ij} &= q_i^+ q_j^- - q_j^+ q_i^- \,, \\
q_{ijk} q_i &= q_{ij}^+ q_{ik}^- - q_{ik}^+ q_{ij}^- \,.
\end{aligned}
$$



$$
Y_{1,1} = -\sqrt{\frac{R^{(+)}}{R^{(-)}}\frac{B^{(-)}}{R^{(+)}}}\,\frac{\mathcal{T}_{1,2}}{T_{2,1}}\left(\frac{T_{1,0}}{Q^+Q^-}\right)^{1+\check{*}\mathcal{Z}}\left(\frac{Q^2}{T_{0,0}}\right)^{*\frac{1}{2}(\mathcal{Z}_1+\mathcal{K}_1)}\left(\frac{T_{1,1}}{\mathcal{T}_{1,1}}\right)^{*\mathcal{K}_1} .
$$

$$
U^2 = \frac{\Lambda^2 T_{00}^-}{\hat{x}^{L-2}Y_{1,1}Y_{2,2}T_{1,0}}\left(\frac{Y_{1,1}Y_{2,2}-1}{\rho/\mathcal{F}^+}\right)^{\hat{*}2\mathcal{Z}}\left(\frac{T_{2,1}\mathcal{T}_{1,1}^-}{\hat{T}_{1,1}^-\mathcal{T}_{1,2}Y_{2,2}}\right)^{*2\Psi}
$$

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