# Integrable systems: solved and open problems in condensed matter physics

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# Contents

- Heisenberg spin chain (susy $tJ$, Hubbard and others ... not today)
- magnetic susceptibility, thermal and spin transport

  Drude weight at zero frequency in dynamical conductivity

- thermodynamics of Heisenberg spin chain: 3 different, but equivalent sets of NLIEs

  quantum chain $\leftrightarrow$ 2d vertex model

  model & mirror model, $Y$-system, $A$-system, (mixed), single $T$ system,...

- Correlations for spin 1/2 (spin-1 and higher ... not today)

  factorization of equal time correlators

  inhomogeneous systems on semi-infinite cylinders

  (i) $L = \infty, T \geq 0$ and (ii) mirror model: $L$ finite, $T = 0$

collaborators: B. Aufgebauer, H. Boos, F. Göhmann, D. Nawrath, J. Suzuki

*Support by Volkswagen Foundation*

Spin-1/2 Hamiltonian

$$H = \sum_{k=1}^{L} \left( S_k^x S_{k+1}^x + S_k^y S_{k+1}^y + \Delta S_k^z S_{k+1}^z \right)$$

| | |
|---|---|
| $-1 < \Delta < 1$ | critical phase (parameterization $\Delta = \cos\gamma$) |
| $\Delta < -1$ | gapped ferromagnetic phase |
| $1 < \Delta$ | gapped antiferromagnetic phase |

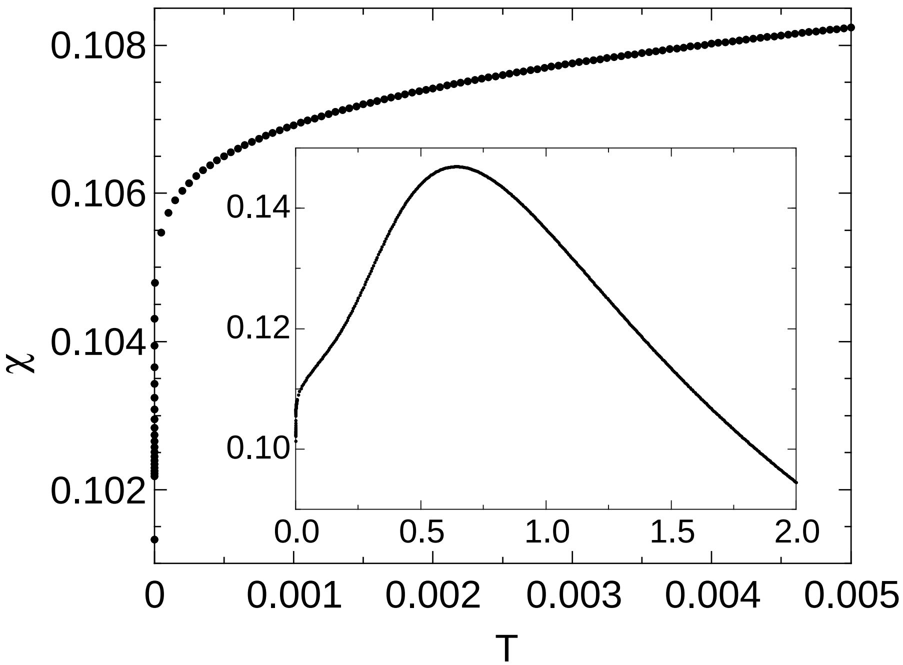
Formulation as lattice gas of spinless fermions

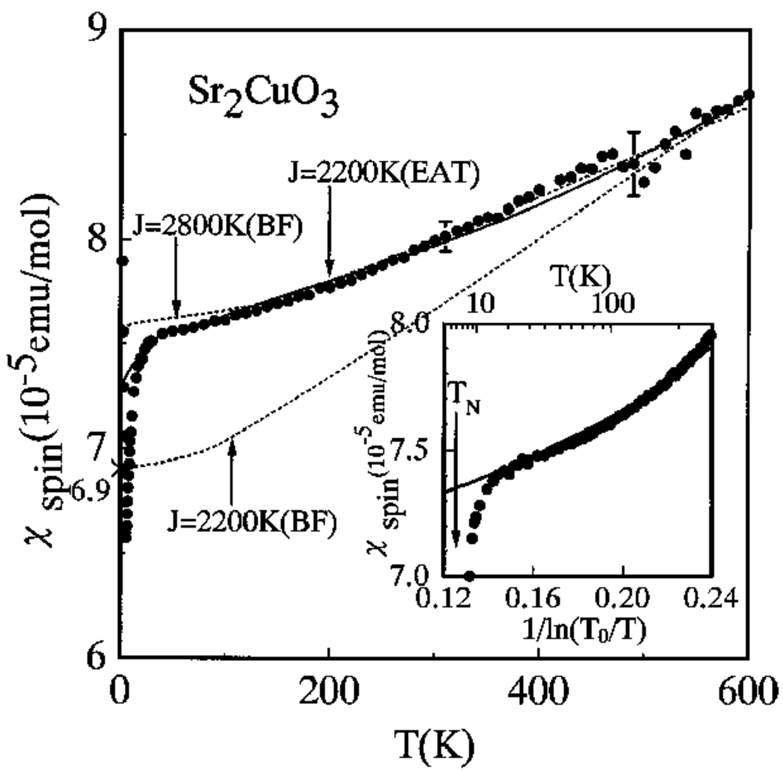
$$H = \sum_{k=1}^{L} \left( c_k^\dagger c_{k+1} + c_{k+1}^\dagger c_k \right) + 2\Delta \sum_{k=1}^{L} n_k n_{k+1}$$

# Heisenberg Chain: Thermodynamics Data

Heisenberg spin-1/2 chain: $H = \sum_j \vec{S}_j \vec{S}_{j+1}$

Susceptibility of isotropic case down to $T/J = 10^{-24}$ and spin chain compound $Sr_2CuO_3$



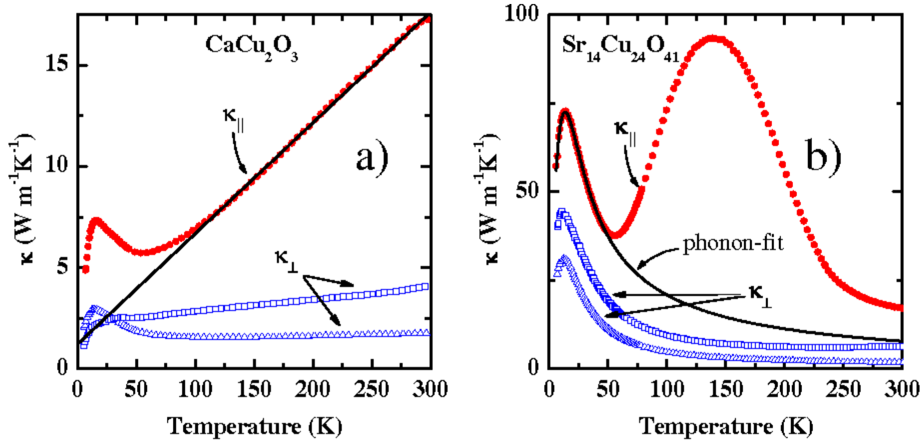




BF: Bonner, Fisher 64; EAT: Eggert, Affleck, Takahashi 94; Motoyama, Eisaki, Uchida 96

AK, Johnston 2000 (low-$T$ and high-$T$ data) $\leftrightarrow$ Lukyanov 98

# Heisenberg Chain: Thermal Conductivities I

Thermal conductivity $\kappa$ relates thermal current $\mathcal{J}_{\mathrm{E}}$ to temperature gradient $\nabla T$: $\mathcal{J}_{\mathrm{E}} = \kappa \nabla T$
Chain and ladder compounds




Phenomenology I: Frequently used model for thermal conductivity: $\kappa = cvl$
($c$ specific heat, $v$ velocity of elementary excitations, $l$ mean free path $= v\tau$)

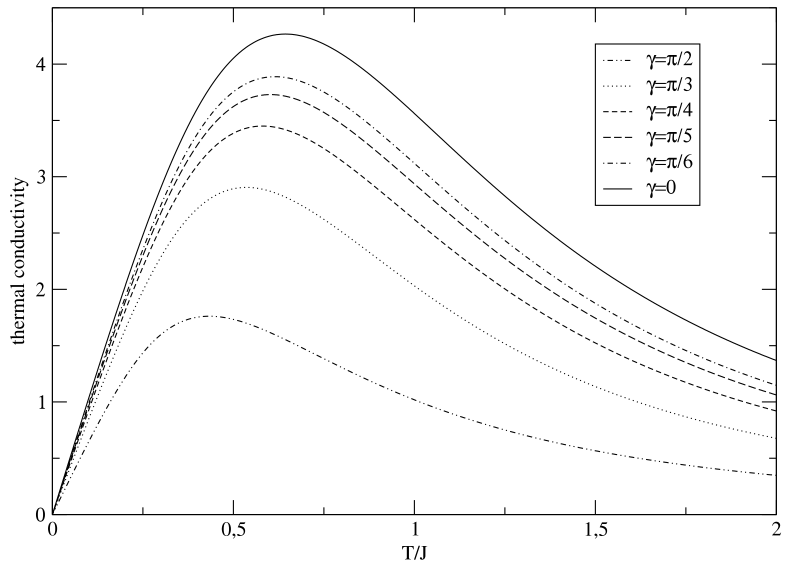
**Kubo Theory** Calculation from 2-point-functions $\kappa(\omega) = \beta \int_0^\infty dt\, e^{-i\omega t} \int_0^\beta d\tau \langle \mathcal{J}_{\mathrm{E}}(-t - i\tau)\mathcal{J}_{\mathrm{E}}\rangle$

Simplification for Heisenberg chain: current is conserved $[\mathcal{H}, \mathcal{J}_{\mathrm{E}}] = 0$:

$$\kappa(\omega) = \frac{1}{i(\omega - i\varepsilon)} \beta^2 \langle \mathcal{J}_{\mathrm{E}}^2 \rangle, \qquad (\varepsilon \to 0+) \qquad \text{especially} \quad \mathrm{Re}\,\kappa(\omega) = \pi D_{\mathrm{th}} \delta(\omega) \qquad D_{\mathrm{th}} = \beta^2 \langle \mathcal{J}_{\mathrm{E}}^2 \rangle$$

Thermal conductivity of $XXZ$ chain at zero frequency is infinite!

Thermal Drude weight $D_{\text{th}}$ (in units of $J^2$)




(critical, short range order) $\Delta = \cos\gamma = 0,\ 0.5,\ 0.707,\ 0.809,\ 0.866,\ 1$ AK, K. Sakai 2002

Phenomenology II: Finite life-time $\tau$ of elementary excitations $\rightarrow$ finite conductivity $\kappa = D_{\text{th}}\tau$

phenomenological pictures I $(\kappa = cvl = cv^2\tau)$ and II consistent if $\dfrac{D_{\text{th}}}{c} = v^2$, is actually satisfied (see above...)

Wiedemann-Franz law $\quad D_{\text{th}}/D_{\text{s}} \simeq \frac{2}{3}\pi(\pi - \gamma)T$

# Thermal current

expression for $j^{\mathrm{E}}$ from continuity equation

$$\frac{\partial}{\partial t} h = -\mathrm{div}\, j^{\mathrm{E}}$$

on lattice: difference equation satisfied with

$$j_k^{\mathrm{E}} = \mathrm{i}[h_{k-1k}, h_{kk+1}]$$

Lüscher 76, Tsvelik 90, Frahm 92, Grabowski et al. 94/95, Zotos et al. 97, Rácz 00

Heisenberg chain: energy current conserved

# Thermodynamics I: TBA *or* $Y$-system

Thermodynamical Bethe Ansatz (Yang+Yang 69, Gaudin 71, Takahashi 71):

combinatorial, free energy functional leading to $\infty$-many auxiliary functions $Y_j,\ j = 1,2,3\ldots$

$$\begin{aligned} \ln Y_1(v) &= -\beta \frac{\frac{\pi}{2}}{\cosh \frac{\pi}{2} v} + \mathbf{s} * \ln(1+Y_2) \\ \ln Y_j(v) &= \mathbf{s} * [\ln(1+Y_{j-1}) + \ln(1+Y_{j+1})], \quad j \geq 2 \end{aligned}$$

where $*$ denotes convolutions and $\mathbf{s}$ is the function

$$\mathbf{s}(v) := \frac{1}{4\cosh \pi v/2}.$$

asymptotical behaviour

$$\lim_{j\to\infty} \frac{\ln Y_j(v)}{j} = \beta h$$

free energy per lattice site

$$\beta f = \beta e - \int_{-\infty}^{\infty} \mathbf{s}(v) \ln(1+Y_1(v)) dv.$$

From TBA to $Y$-system: Zamolodchikov 90; from $T$-system to $Y$-system and TBA: AK, Pearce 92

# Thermodynamics II: Spinon formulation *or* $A$-system

two auxiliary functions $\mathfrak{a}$, $\overline{\mathfrak{a}}$

$$\begin{aligned}
\log \mathfrak{a}(v) &= +\frac{\beta h}{2} - \beta e(v+\mathrm{i}) + \kappa * [\log(1+\mathfrak{a}) - \log(1+\overline{\mathfrak{a}})], \\
\log \overline{\mathfrak{a}}(v) &= -\frac{\beta h}{2} - \beta e(v-\mathrm{i}) + \kappa * [\log(1+\overline{\mathfrak{a}}) - \log(1+\mathfrak{a})].
\end{aligned}$$

where $e(v)$ and the kernel $\kappa(v)$ take the form

$$e(v) := \frac{\frac{\pi}{2}}{\cosh \frac{\pi}{2} v}, \qquad \kappa(v) := \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{\mathrm{e}^{-|k|}}{\mathrm{e}^{k} + \mathrm{e}^{-k}} \mathrm{e}^{\mathrm{i}kv} dk$$

free energy

$$\beta f = \beta e_0 - \frac{1}{2\pi} \int_{-\infty}^{\infty} e(v) \log[(1+\mathfrak{a}(v-\mathrm{i}))(1+\overline{\mathfrak{a}}(v+\mathrm{i}))] dv,$$

AK, Batchelor 90; AK, Batchelor, Pearce 91; AK 92,93; Destri, de Vega 92,95; J. Suzuki 98, Hegedűs 05

general $sl(N)$ etc./Bäcklund transformations: ...; Zabrodin 07; Arutyunov, Frolov 09; Tsuboi 11; Kazakov, Leurent, Tsuboi 12; Frolov, Quinn 12; Balog, Hegedűs 12

# Thermodynamics III: one integral equation *or* single $T$-system

free energy

$$\beta f = -\log T(0)$$

where $u(v)$ satisfies

$$T(v) = 2\cosh(\beta h/2) + \frac{1}{2\pi i}\oint_C \left[\frac{b(w+\mathrm{i})}{v-2\mathrm{i}-w} + \frac{b(w-\mathrm{i})}{v+2\mathrm{i}-w}\right]\frac{1}{T(w)}\mathrm{d}w$$

and $b(v)$ explicitly known

$$b(v) = \exp\left(\mathrm{i}\frac{\beta}{v+\mathrm{i}} - \mathrm{i}\frac{\beta}{v-\mathrm{i}}\right)$$

Takahashi cond-mat/0010486; Takahashi, Shiroishi, AK 01; Tsuboi 04/05, ...

# Note on generalizations

- NLIE for eigenvalues of Hamiltonians: same as those for transfer matrix if $\beta e(v) \leftrightarrow Lp(v)$

- Excited states by deformation of contours or by additional driving terms

AK, Pearce 91: Tricritical hard squares and critical hard hexagons

AK, Pearce 92, 93: Conformal weights of RSOS lattice models (analytic continuation)

AK, Wehner, Zittartz 93: Conformal spectrum of the six-vertex model by dilogarithm-trick

$$x = \frac{1-\gamma/\pi}{2} S^2 + \frac{1}{2(1-\gamma/\pi)} m^2$$

Dorey, Tateo 96, 98: Excited states by analytic continuation of TBA equations

Fioravanti, Mariottini, Quattrini, Ravanini 97: Excited States for (Restricted) Sine-Gordon Models

Fabricius, AK, McCoy 99: Temperature dependent spatial oscillations...

AK, Reyes Martínez, Scheeren, Shiroishi 01: $XXZ$ chain at finite magnetic field...

Bajnok, Hegedűs, Janik, Łukowski 09: Five loop Konishi from AdS/CFT

Balog, Hegedűs 09: Finite size spectrum of 2d $O(3)$ nl$\sigma$

Arutyunov, Frolov, R. Suzuki 10: Exploring the mirror TBA

Gromov, Kazakov, Kozak, Vieira 10: Anomalous dimensions of planar $N = 4$ susy YM

# Spin current

The spin current density

$$j_k^{\rm s} = {\rm i}\left(S_k^+ S_{k+1}^- - S_k^- S_{k+1}^+\right)$$

and its the continuity equation

$$\frac{\partial}{\partial t} S^z = -{\rm div}\, j^{\rm s}.$$

Total spin current $\mathcal{J}_{\rm s} = \sum_k j_k^{\rm s}$ is not conserved (except for $\Delta = 0$, free fermions)

$$[\mathcal{H}, \mathcal{J}] \neq 0, \qquad D(T) \neq \beta \langle \mathcal{J}^2 \rangle.$$

However: Drude weight is related to energy level curvature with respect to twisted boundary conditions

$$D = \frac{1}{2L} \sum_n p_n \frac{\partial^2 \varepsilon_n[\Phi]}{\partial \Phi^2}\bigg|_{\Phi=0}$$ Kohn 1964,...

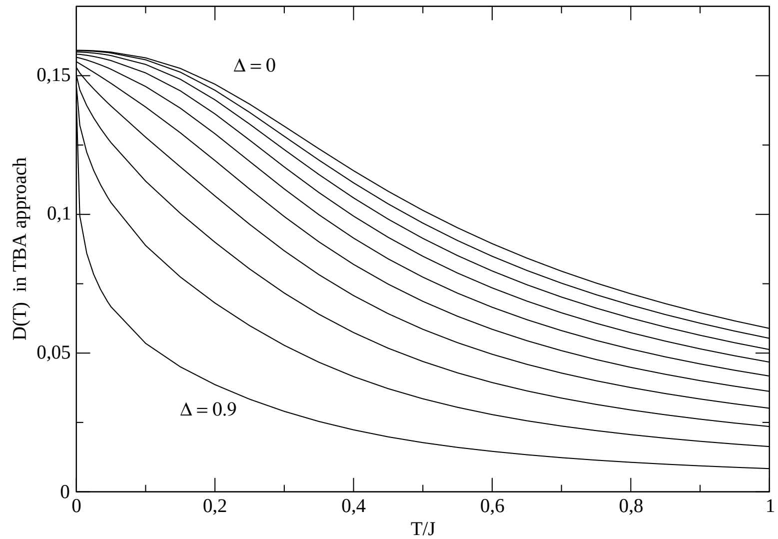
$T = 0$: The groundstate energy curvature is analytically known (Shastry, Sutherland 90)

$$D_{\rm s}(T=0) = \frac{v}{2(\pi - \gamma)}$$ where $\Delta = \cos\gamma$.

$T > 0$: TBA-like computational scheme by Fujimoto, Kawakami 98; application to $XXZ$: Zotos 98

# Extended Thermodynamical Bethe ansatz

Thermodynamical Bethe ansatz applied to magnons and their bound states




Zotos 98

Thermodynamical Bethe ansatz based on:

- magnons and their bound states
- scattering of these 'particles'
- construction of scattering states
- minimization of free energy functional

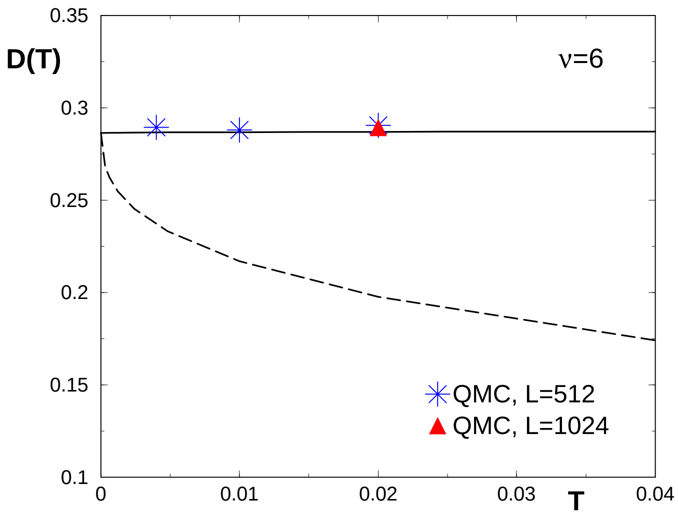
Problems for calculating energy curvatures for large but finite chain length:

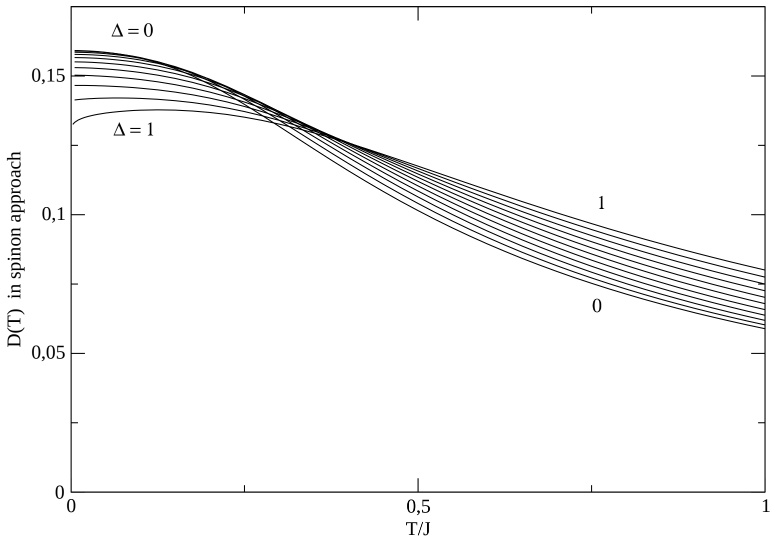
- composition of bound magnon states (assumption in TBA: 'ideal strings')
- distribution of these states does not follow continuous density distribution (but assumed in TBA)

# QMC and TBA-like approach

Quantum Monte Carlo

TBA on spinon basis




Alvarez, Gros 02

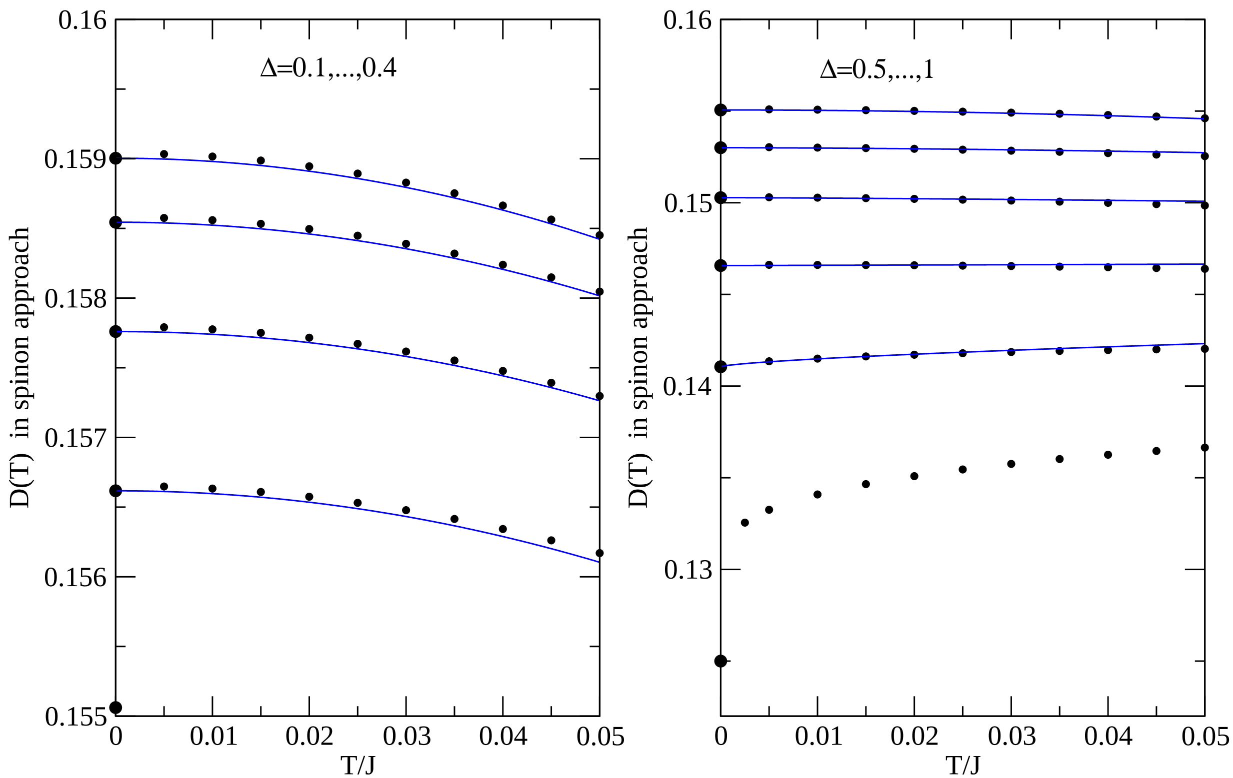
Benz, Fukui, AK, Scheeren (01, 05)

finite chains: Heidrich-Meisner et al. 03; improved QMC Brenig, Grossjohann 10

Extended thermodynamical Bethe ansatz (non-linear integral equation for $a$ and 'conjugate' equation for $\bar{a}$):

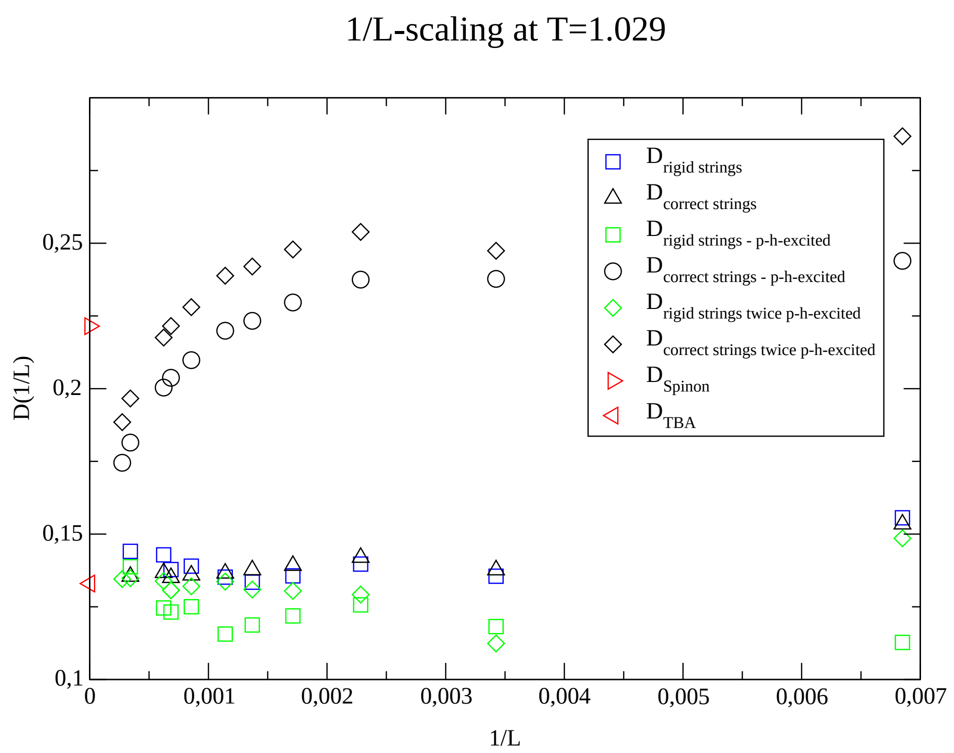
$$D = \frac{T}{4\pi}\left[\int_{-\infty}^{\infty} dx \frac{\left(\frac{\partial a}{\partial h}\cdot\frac{\partial a}{\partial x}\right)^2}{a^2(1+a)^2\frac{\partial a}{\partial T}} + (a \leftrightarrow \bar{a})\right], \qquad \log a(x) = +\frac{\beta h}{2} - \beta e(x) + \kappa * [\log(1+a) - \log(1+\bar{a})]$$

# Conformal perturbation theory




Comparison of spinon-TBA results with CFT data by J. Sirker(05,09)
CFT with band curvature, no umklapp

However, real time dynamics by TMRG etc. suggest $D(T) = 0$ for $XXZ$ chain with $h = 0$ (Sirker et al. 09).
But very recent numerics...




Glocke, AK 02 unpublished

curvature depends on state!

just in (wrong) rigid-string picture: all curvatures same

- summation over all states necessary!
- case $T > 0$ qualitatively different from $T = 0$! All microstates for $T = 0$ (low-lying excitations):

$$E_x(\varphi) - E_{g.s.}(0) = \frac{2\pi}{L} v x + o\left(1/L\right), \qquad x(\varphi) = x_{S,m}(\varphi) = \frac{1-\gamma/\pi}{2} S^2 + \frac{1}{2(1-\gamma/\pi)} \left(m - \frac{\varphi}{\pi}\right)^2$$

# Proper calculation of curvatures by use of integrability

Problem of calculation by use of Bethe ansatz equations

- bound magnon states: single magnon rapidities close to poles of scattering phases (singular)
- continuous density distributions not sufficient

Alternative approach to Bethe ansatz: 'fusion algebra'
for $\Delta = \cos\frac{\pi}{\nu}$ particularly simple: finite number of functions

$$\log Y_1(v) = L \log \mathrm{th} \tfrac{\pi}{4} v + \sum_{\zeta_2} \log \mathrm{th} \frac{\pi}{4}(v - \zeta_2) + s * \log(1 + Y_2)$$

$$\log Y_j(v) = + \sum_{\zeta_{j-1}} \log \mathrm{th} \frac{\pi}{4}(v - \zeta_{j-1}) + \sum_{\zeta_{j+1}} \log \mathrm{th} \frac{\pi}{4}(v - \zeta_{j+1}) + s * \log[(1 + Y_{j-1})(1 + Y_{j+1})]$$

$$\log Y_{\nu-2}(v) = + \sum_{\zeta_{\nu-3}} \log \mathrm{th} \frac{\pi}{4}(v - \zeta_{\nu-3}) + \sum_{\zeta_\pm} \log \mathrm{th} \frac{\pi}{4}(v - \zeta_\pm) + s * \log[(1 + Y_{\nu-3})(1 + Y_+)(1 + Y_-)]$$

$$\log Y_\pm(v) = \pm i\phi + \sum_{\zeta_{\nu-2}} \log \mathrm{th} \frac{\pi}{4}(v - \zeta_{\nu-2}) + s * \log(1 + Y_{\nu-2})$$

(A. Kuniba, K. Sakai, J. Suzuki 98)

where $*$ denotes convolution and $\mathbf{s}(v) := \frac{1}{4\cosh \pi v/2}$, energy $E = \sum_{\zeta_1} \frac{\pi/2}{\cosh \pi\zeta_1/2} + \int_{-\infty}^{\infty} \frac{\log(1+Y_1)(x)}{\sinh \pi x/2} dx$

dropping of all integrals $\rightarrow$ Bethe ansatz equations for bound magnon states

# Integrable $su(2)$ quantum chains: $S = 1/2$ case

Heisenberg $S = 1/2$ chain

$$H = \sum_j \vec{S}_j \vec{S}_{j+1}$$

Near-neighbor correlators ($L = \infty$, $T = 0$):

$$\begin{aligned}
\left\langle S^z_j S^z_{j+1} \right\rangle &= \frac{1}{12} - \frac{1}{3}\ln 2 = -0.1477157268\ldots \\
\left\langle S^z_j S^z_{j+2} \right\rangle &= \frac{1}{12} - \frac{4}{3}\ln 2 + \frac{3}{4}\zeta(3) = 0.06067976995\ldots \\
\left\langle S^z_j S^z_{j+3} \right\rangle &= \frac{1}{12} - 3\ln 2 + \frac{37}{6}\zeta(3) - \frac{14}{3}\ln 2 \cdot \zeta(3) - \frac{3}{2}\zeta(3)^2 - \frac{125}{24}\zeta(5) + \frac{25}{3}\ln 2 \cdot \zeta(5) \\
&= -0.05024862725\ldots
\end{aligned}$$

Hulthén (1938), Takahashi (1977): ground state energies of Heisenberg and Hubbard model

Multiple integral formulas for density matrix:

Vertex operator approach: Jimbo, Miki, Miwa, Nakayashiki (92); qKZ equation Jimbo, Miwa (96)

Boos, Korepin (01/02)

$T = 0$, $h \geq 0$ Kitanine, Maillet, Terras (1998); $T \geq 0$, $h \geq 0$ Göhmann, AK, Seel (2004).

# $su(2)$ $S = 1/2$: 2-point correlators

- analytic results show feasibility of exact calculations
- the smaller the numerical results the larger the analytical expressions

$$
\begin{aligned}
\left\langle S_j^z S_{j+4}^z \right\rangle &= \frac{1}{12} - \frac{16}{3}\ln 2 + \frac{145}{6}\zeta(3) - 54\ln 2\cdot\zeta(3) - \frac{293}{4}\zeta(3)^2 - \frac{875}{12}\zeta(5) + \frac{1450}{3}\ln 2\cdot\zeta(5) \\
&\quad - \frac{275}{16}\zeta(3)\cdot\zeta(5) - \frac{1875}{16}\zeta(5)^2 + \frac{3185}{64}\zeta(7) - \frac{1715}{4}\ln 2\cdot\zeta(7) + \frac{6615}{32}\zeta(3)\cdot\zeta(7) \\
&= 0.034652776982\ldots \\
\left\langle S_j^z S_{j+5}^z \right\rangle &= \text{7 lines} \\
\left\langle S_j^z S_{j+6}^z \right\rangle &= \text{18 lines (= 1 page)} \\
\left\langle S_j^z S_{j+7}^z \right\rangle &= \text{3 pages}
\end{aligned}
$$

# $su(2)$ $S = 1/2$: more correlators

- occurence of zeta-function values seem to indicate that no algebraic structure exists

  no algebraic relation of 2-point functions for different separation

  no algebraic, *Wick theorem*-like relation of general $n$-site correlations

$$
\begin{aligned}
\left\langle S_j^z S_{j+1}^z \right\rangle &= \frac{1}{12} - \frac{1}{3}\ln 2 \\
\left\langle S_j^z S_{j+2}^z \right\rangle &= \frac{1}{12} - \frac{4}{3}\ln 2 + \frac{3}{4}\zeta(3) \\
\left\langle S_j^z S_{j+3}^z \right\rangle &= \frac{1}{12} - 3\ln 2 + \frac{37}{6}\zeta(3) - \frac{14}{3}\ln 2\cdot\zeta(3) - \frac{3}{2}\zeta(3)^2 - \frac{125}{24}\zeta(5) + \frac{25}{3}\ln 2\cdot\zeta(5) \\
\left\langle S_j^x S_{j+1}^x S_{j+2}^z S_{j+3}^z \right\rangle &= \frac{1}{240} + \frac{\ln 2}{12} - \frac{91}{240}\zeta(3) + \frac{1}{6}\ln 2\cdot\zeta(3) + \frac{3}{80}\zeta(3)^2 + \frac{35}{96}\zeta(5) - \frac{5}{24}\ln 2\cdot\zeta(5), \\
\left\langle S_j^x S_{j+1}^z S_{j+2}^x S_{j+3}^z \right\rangle &= \frac{1}{240} - \frac{\ln 2}{6} + \frac{77}{120}\zeta(3) - \frac{5}{12}\ln 2\cdot\zeta(3) - \frac{3}{20}\zeta(3)^2 - \frac{65}{96}\zeta(5) + \frac{5}{6}\ln 2\cdot\zeta(5), \\
\left\langle S_j^x S_{j+1}^z S_{j+2}^z S_{j+3}^x \right\rangle &= \frac{1}{240} - \frac{\ln 2}{4} + \frac{169}{240}\zeta(3) - \frac{5}{12}\ln 2\cdot\zeta(3) - \frac{3}{20}\zeta(3)^2 - \frac{65}{96}\zeta(5) + \frac{5}{6}\ln 2\cdot\zeta(5)
\end{aligned}
$$

# $su(2)$ $S=1/2$: correlators of inhomogeneous system

Another example of correlators: emptiness formation probability (Boos, Korepin, Smirnov 03)

$$P(n) := \left\langle \left(S_1^z + \frac{1}{2}\right)\left(S_2^z + \frac{1}{2}\right)\cdots\left(S_n^z + \frac{1}{2}\right)\right\rangle = (D_n)^{++\dots+}_{++\dots+}$$

Explicitly for spin-1/2 Heisenberg ($T=0$)

$$P(1) = \frac{1}{2}, \qquad P(2) = \frac{1}{3} - \frac{1}{3}\ln 2, \qquad P(3) = \frac{1}{4} - \ln 2 + \frac{3}{8}\zeta(3)$$

$$P(4) = \frac{1}{5} - 2\ln 2 + \frac{173}{60}\zeta(3) - \frac{11}{6}\ln 2\cdot\zeta(3) - \frac{51}{80}\zeta^2(3) - \frac{55}{24}\zeta(5) + \frac{85}{24}\ln 2\cdot\zeta(5).$$

Inhomogeneous generalization of correlators enjoys more algebraic relations ($\lambda_{ij} := \lambda_i - \lambda_j$)

$$P(1) = \frac{1}{2}, \quad P(2) = \frac{1}{4} + \frac{1}{6}\omega(\lambda_1,\lambda_2), \quad P(3) = \frac{1}{8} + \frac{1+\lambda_{13}\lambda_{23}}{12\lambda_{13}\lambda_{23}}\omega(\lambda_1,\lambda_2) \quad \text{(+ 2 permutations)}$$

$$P(4) = \frac{1}{16} + \frac{5(1+\lambda_{14}\lambda_{24})(1+\lambda_{13}\lambda_{23}) - (\lambda_{12}^2 - 4)}{120\lambda_{13}\lambda_{23}\lambda_{14}\lambda_{24}}\omega(\lambda_1,\lambda_2) \quad \text{(+ 5 permutations)}$$

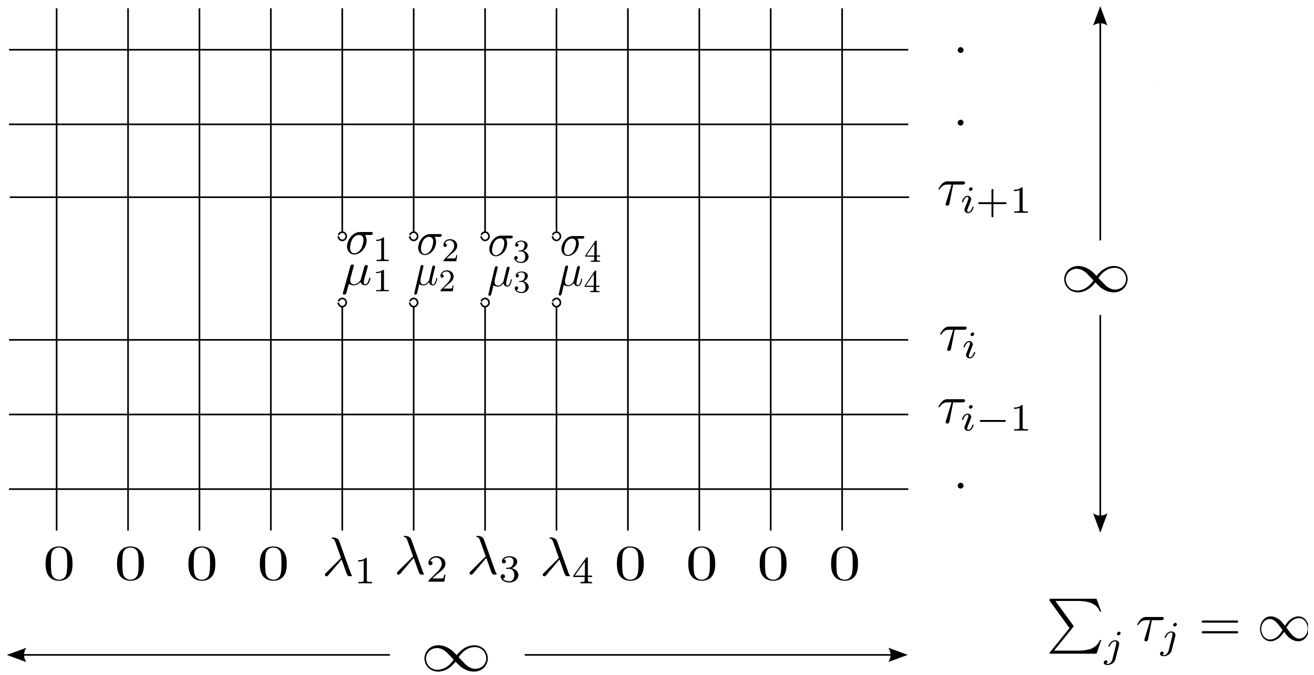
$$+\frac{15 + 10(\lambda_{13}\lambda_{24}+1)(1+\lambda_{14}\lambda_{23}) + (2\lambda_{12}^2 - 3)(2\lambda_{34}^2 - 3)}{360\lambda_{13}\lambda_{23}\lambda_{14}\lambda_{24}}\omega(\lambda_1,\lambda_2)\omega(\lambda_3,\lambda_4)$$

(+ 2 permutations)

# Quantum chains, classical systems and density matrix

Mapping 1d quantum system to 2d classical allows for generalization of $D_n$ as meromorphic function $D_n(\lambda_1, \lambda_2, ..., \lambda_n)$.

Unnormalized matrix element $D^{\mu_1,\ldots,\mu_4}_{\sigma_1,\ldots,\sigma_4}$ given by partition function of some six-vertex model:




The $\omega(\lambda_1, \lambda_2)$ function is given by the nearest-neighbour correlator

$$(T = h = 0) \qquad \omega(\lambda_1, \lambda_2) := \frac{1}{2} + 2\sum_{k=1}^{\infty}(-1)^k k \frac{1-x^2}{k^2-x^2}, \qquad x := \lambda_1 - \lambda_2.$$

# $su(2)$ $S = 1/2$: factorization

Various questions arise when looking at expressions for correlators like ($\lambda_{ij} := \lambda_i - \lambda_j$)

$$P(1) = \frac{1}{2}, \quad P(2) = \frac{1}{4} + \frac{1}{6}\omega(\lambda_1, \lambda_2), \quad P(3) = \frac{1}{8} + \frac{1 + \lambda_{13}\lambda_{23}}{12\lambda_{13}\lambda_{23}}\omega(\lambda_1, \lambda_2) \quad \text{(+ 2 permutations)}$$

$$\begin{aligned} P(4) &= \frac{1}{16} + \frac{5(1+\lambda_{14}\lambda_{24})(1+\lambda_{13}\lambda_{23}) - (\lambda_{12}^2 - 4)}{120\lambda_{13}\lambda_{23}\lambda_{14}\lambda_{24}}\omega(\lambda_1, \lambda_2) \qquad \text{(+ 5 permutations)} \\ &+ \frac{15 + 10(\lambda_{13}\lambda_{24}+1)(1+\lambda_{14}\lambda_{23}) + (2\lambda_{12}^2 - 3)(2\lambda_{34}^2 - 3)}{360\lambda_{13}\lambda_{23}\lambda_{14}\lambda_{24}}\omega(\lambda_1, \lambda_2)\omega(\lambda_3, \lambda_4) \\ &\qquad \text{(+ 2 permutations)} \end{aligned}$$

- How do these expressions appear? (Multiple integral expressions, functional equations)
- Is the limit $\lambda_1, \lambda_2, \lambda_3, \lambda_4 \to 0$ singular?

  algebraically: yes $\longrightarrow$ therefore no algebraic structure in homogeneous limit

  analytically: no $\longrightarrow$ therefore nothing wrong about it
- Why should we care about such factorized expressions for correlators of inhomogenous systems?

# $su(2)$ $S=1/2$: Finite temperatures

• Why should we care about such factorized expressions for correlators of inhomogenous systems?

We care, because we can prove the (literally) same factorization for arbitrary temperature...

... and we can calculate $\omega(\lambda_1,\lambda_2)$!

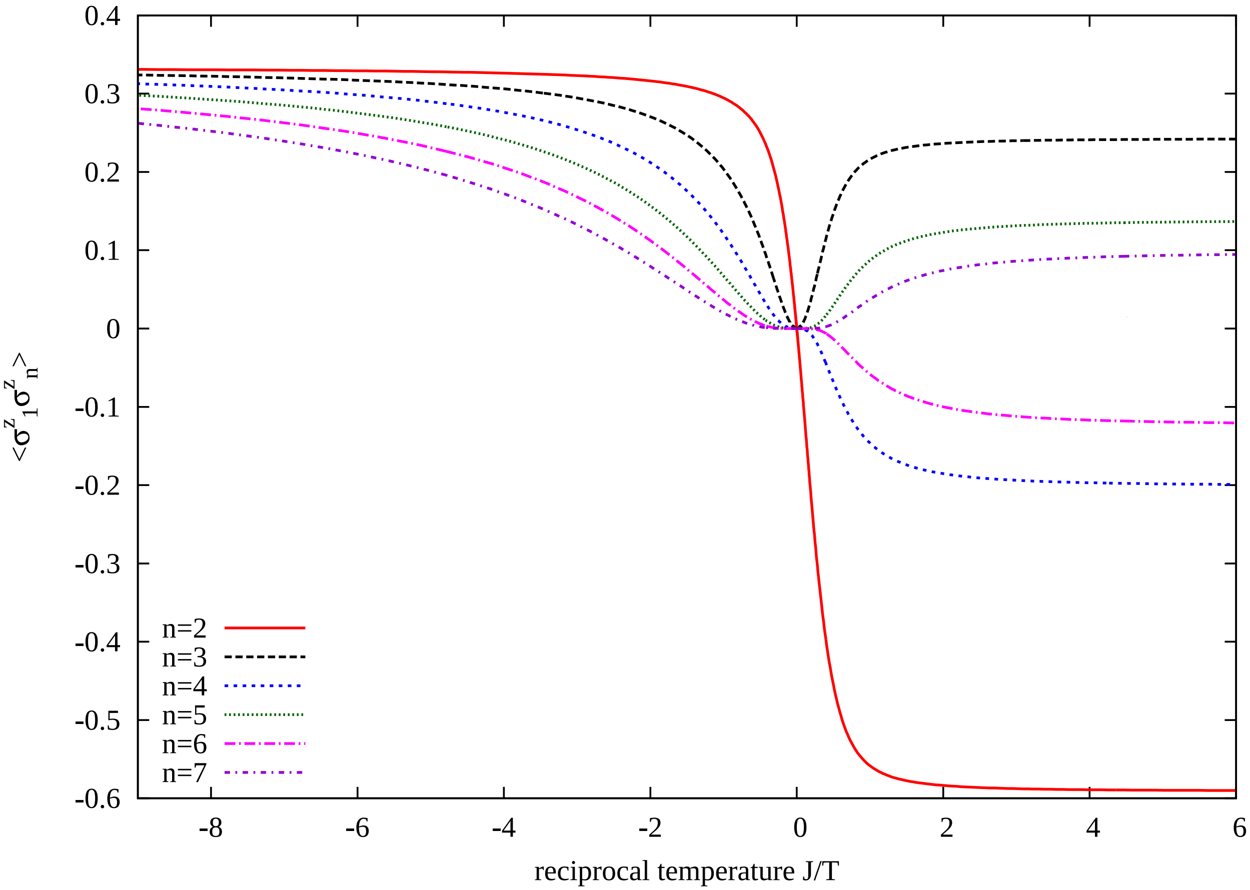
$$\omega(\lambda_1,\lambda_2) := \frac{1}{2} + \frac{(\lambda_1-\lambda_2)^2-1}{2\pi}\int_C \frac{d\mu}{1+a(\mu)}\frac{G(\mu,\mathrm{i}\lambda_1)}{(\mu-\mathrm{i}\lambda_2)(\mu-\mathrm{i}\lambda_2-\mathrm{i})}$$

where the functions $G$ and $a$ satisfy the linear and non-linear integral equations

$$G(\lambda,\lambda_1) = -\frac{1}{(\lambda-\lambda_1)(\lambda-\lambda_1-\mathrm{i})} + \frac{1}{\pi}\int_C \frac{d\mu}{1+a(\mu)}\frac{G(\mu,\lambda_1)}{1+(\lambda-\mu)^2}$$

$$\log a(\lambda) = 2J\frac{\beta}{\lambda(\lambda+\mathrm{i})} - \frac{1}{\pi}\int_C \frac{\log(1+a(\mu))}{1+(\lambda-\mu)^2}d\mu,$$

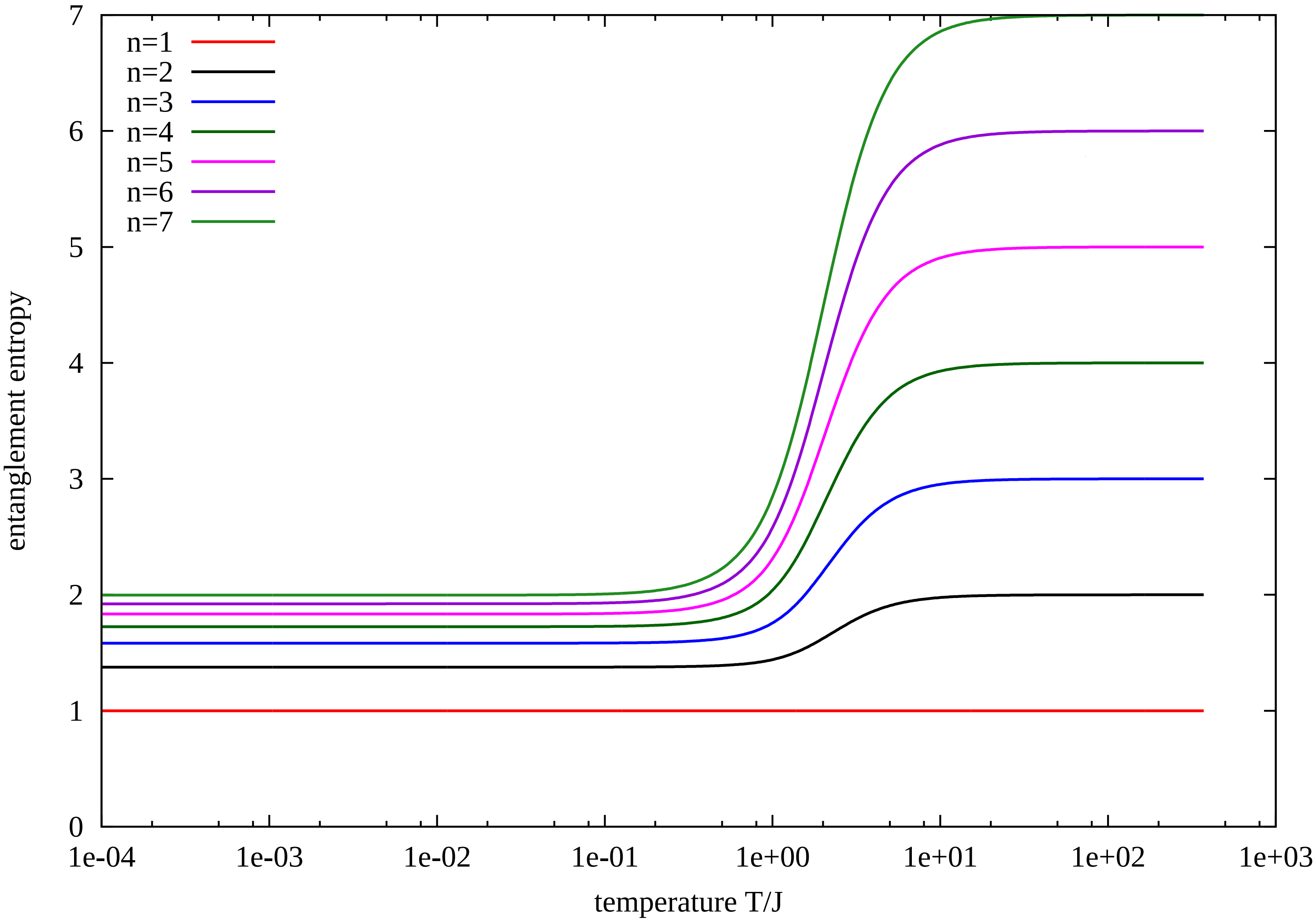
Sato, Aufgebauer, Boos, Göhmann, AK, Takahashi, Trippe (11)




AF: low-temperature behaviour $\langle \sigma_1^z \sigma_{1+r}^z \rangle \simeq \langle \sigma_1^z \sigma_{1+r}^z \rangle_0 (1 - \gamma_r (T/J)^2)$

$\langle \sigma_1^z \sigma_2^z \rangle \simeq \frac{1}{3} - \frac{4}{3} \ln 2 + \frac{1}{36} (T/J)^2$, $\langle \sigma_1^z \sigma_3^z \rangle \simeq \frac{1}{3} - \frac{16}{3} \ln 2 + 3\zeta(3) + \left( \frac{1}{9} - \frac{\pi^2}{72} \right) (T/J)^2$

AF case: entanglement entropy for $n$ successive sites



# Proofs and constructions

representation theory of quantum algebras/vertex operators ($T = h = 0$): Jimbo, Miki, Miwa, Nakayashiki 92; Jimbo, Miwa 95

functional equations, qKZ ($T = h = 0$): Jimbo, Miwa 96

algebraic Bethe ansatz: ($T = 0, h \neq 0$): Kitanine, Maillet, Terras 99
($T \geq 0, h \neq 0$): Göhmann, Klümper, Seel 04, 05; Boos, Göhmann 09

factorization of multiple integrals/correlation functions ($T = h = 0$): Boos, Korepin 01; Boos, Korepin, Smirnov 03, 04; Boos, Jimbo, Miwa, Smirnov, Takeyama 05, 06; Kato, Shiroishi, Takahashi, Sakai 03; Sato, Shiroishi, Takahashi 05

exponential formula for the reduced density matrix ($T = 0$): Boos, Jimbo, Miwa, Smirnov, Takeyama 06 (2 papers)

conjecture of exponential formula for finite temperature/finite length: Boos, Göhmann, Klümper, Suzuki 06; Damerau, Göhmann, Hasenclever, Klümper 07; Boos, Göhmann, Klümper, Suzuki 07

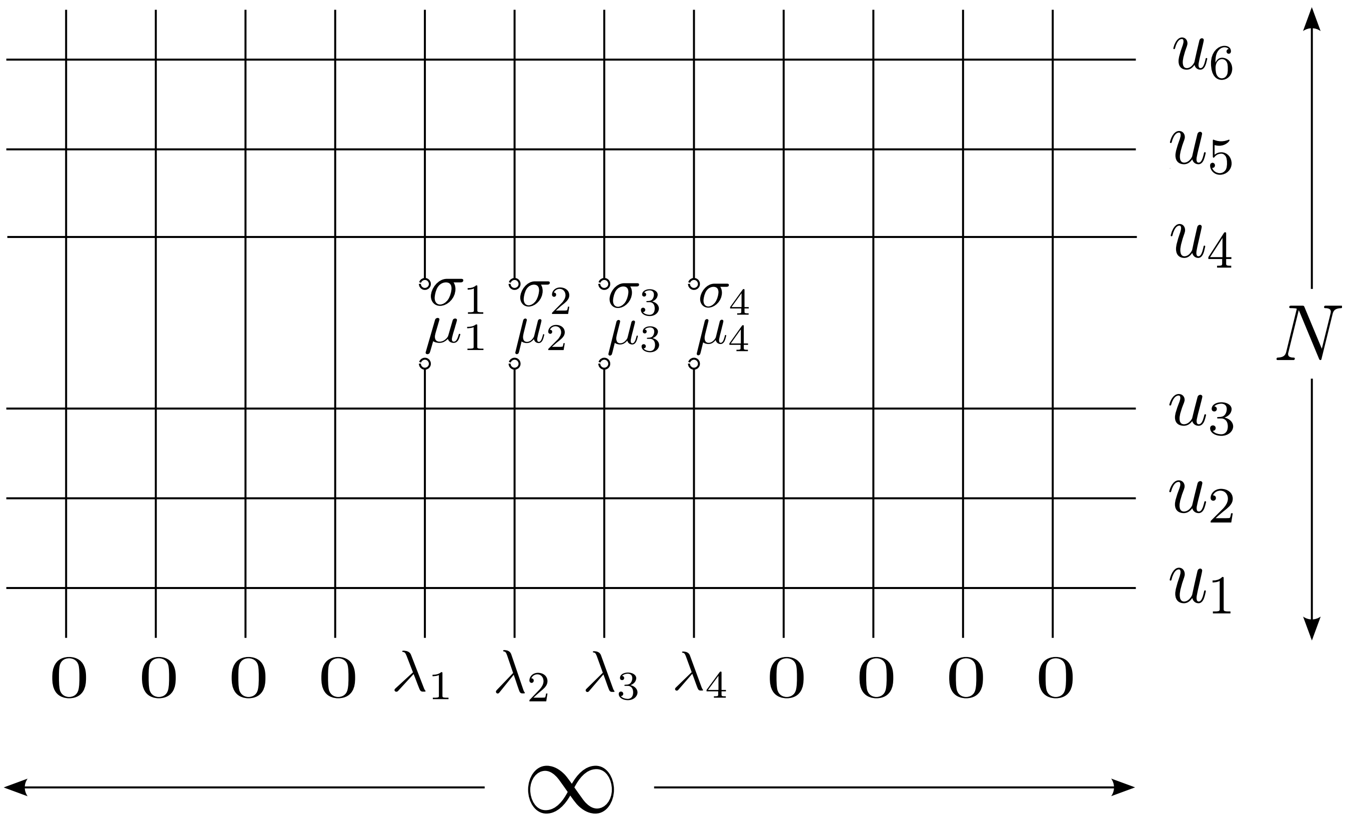
fermionic structure on space of local operators: Boos, Jimbo, Miwa, Smirnov, Takeyama 07, 09

algebraic proof of exponential formula/factorization: Jimbo, Miwa, Smirnov 09

algebraic/analytic proof: Aufgebauer, AK 12

# Correlation functions/reduced density matrix

All correlation functions of a sequence of spin operators on consecutive $n$ sites determined by reduced density matrix $D_n$ with matrix elements $D_n{}^{(\mu_1,\mu_2,\ldots,\mu_n)}_{(\sigma_1,\sigma_2,\ldots,\sigma_n)}$. Finite Trotter number $N$.




$D_n(\lambda_1,\lambda_2,\ldots,\lambda_n)$ is meromorphic: numerator = unknown $n$-variate polynomial of degree $N$

denominator = $\Lambda_0(\lambda_1)\cdot\ldots\cdot\Lambda_0(\lambda_n)$, largest eigenvals of QTM

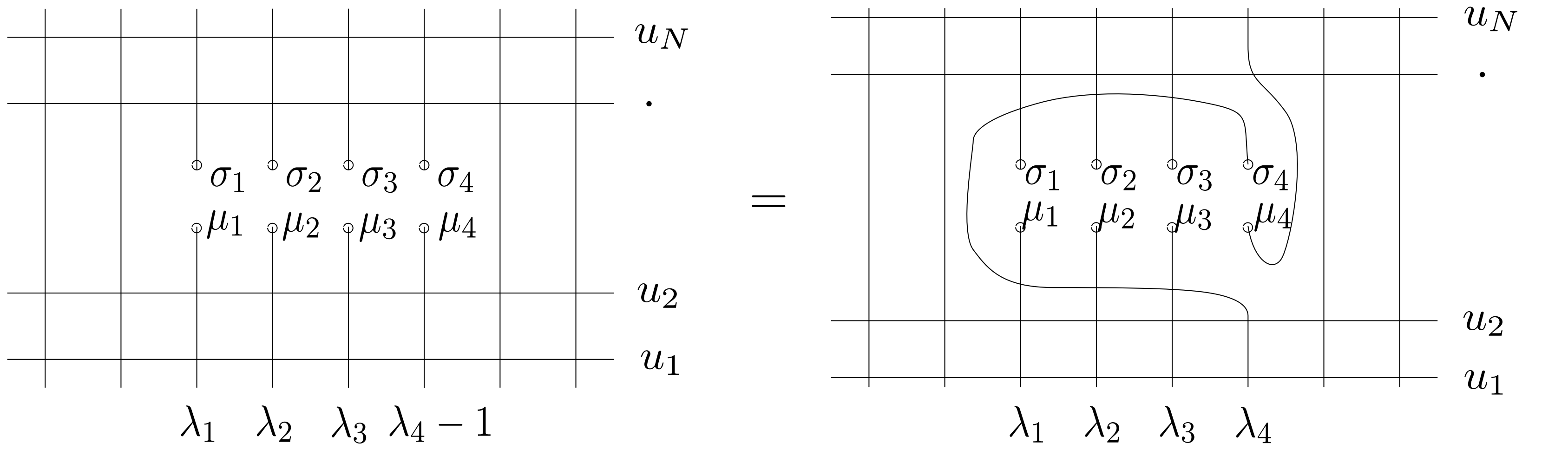
# Discrete functional equation/unique solution

Functional equation ('rqKZ'-equation) for density matrix on $\infty \times N$–lattice

$$D_n(\lambda_1, \ldots, \lambda_{n-1}, \lambda_n - 1) = A(\lambda_1, \ldots, \lambda_n) D_n(\lambda_1, \ldots, \lambda_{n-1}, \lambda_n)$$

for arbitrary complex $\lambda_1, \ldots, \lambda_{n-1}$ and $\lambda_n$ from set of spectral parameters on horizontal lines $\{v_1, v_2, \ldots, v_N\}$. (Sato et al. (11); Aufgebauer, AK (12))

$A$ is a linear operator acting in the space of density matrices, example for $n = 4$:




Discrete functional equation + analyticity conditions + asymptotics fix $D_n$ for arbitrary Trotter number $N$ ($T \geq 0$). (Aufgebauer, AK (12))

# $su(2)$ $S = 1/2$: The mirror model

Mirror model: Change of space and time direction

Change of driving term in NLIE (notice absence of Lorentz invariance)

$$2J \frac{\beta}{\lambda(\lambda + \mathrm{i})} \qquad \Longrightarrow \qquad L \log \frac{\lambda - \mathrm{i}/2}{\lambda + \mathrm{i}/2}$$

yielding

$$\omega(\lambda_1, \lambda_2) := \frac{1}{2} + \frac{(\lambda_1 - \lambda_2)^2 - 1}{2\pi} \int_C \frac{d\mu}{1 + a(\mu)} \frac{G(\mu, \mathrm{i}\lambda_1)}{(\mu - \mathrm{i}\lambda_2)(\mu - \mathrm{i}\lambda_2 - \mathrm{i})}$$

where the functions $G$ and $a$ satisfy the linear and non-linear integral equations

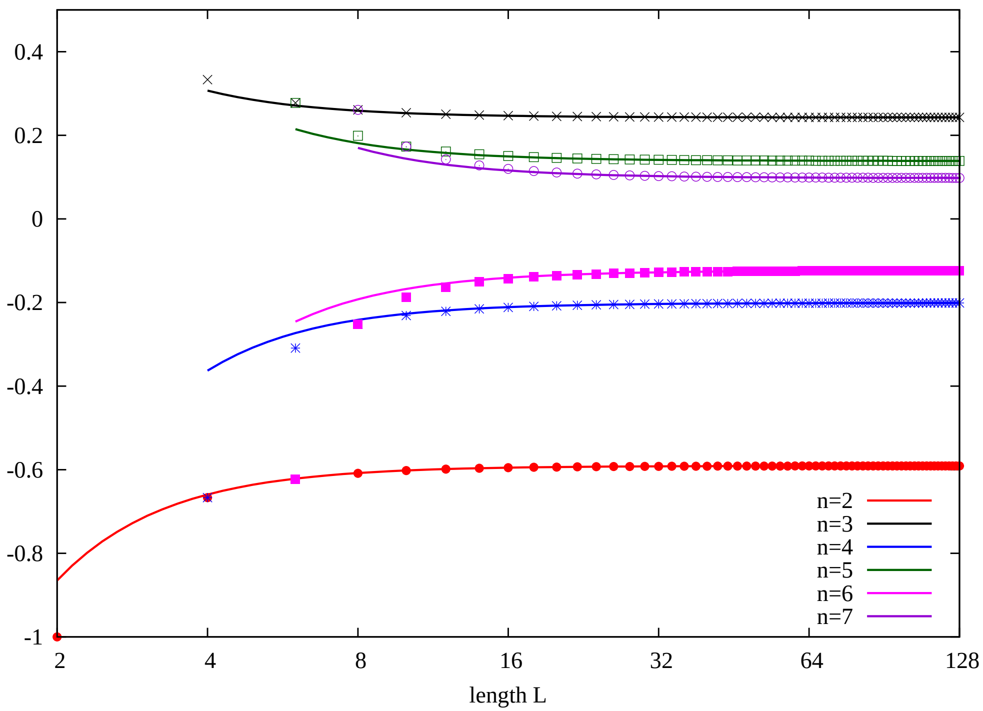
$$G(\lambda, \lambda_1) = -\frac{1}{(\lambda - \lambda_1)(\lambda - \lambda_1 - \mathrm{i})} + \frac{1}{\pi} \int_C \frac{d\mu}{1 + a(\mu)} \frac{G(\mu, \lambda_1)}{1 + (\lambda - \mu)^2}$$

$$\log a(\lambda) = L \log \frac{\lambda - \mathrm{i}/2}{\lambda + \mathrm{i}/2} - \frac{1}{\pi} \int_C \frac{\log(1 + a(\mu))}{1 + (\lambda - \mu)^2} d\mu,$$

Sato, Aufgebauer, Boos, Göhmann, AK, Takahashi, Trippe (11)

Mirror model: Antiferromagnetic Heisenberg chain at $T = 0$ and $L = 2, 4, 6, \ldots$.

2-point correlators $\langle \sigma_1^z \sigma_n^z \rangle$ in dependence on chain length $L$ for different point separations $n$.




AF: finite size scaling $\langle \sigma_1^z \sigma_{1+r}^z \rangle \simeq \langle \sigma_1^z \sigma_{1+r}^z \rangle_0 (1 + 4\gamma_r \pi^2 / L^2)$.

# Summary

Spin-1/2 Heisenberg chain for $T > 0$ and arbitrary $h$

Exact results obtained for

- free energy
- thermal conductivity ($h = 0$)

Open problems

- spin Drude weight

Presentation of algebraic expressions for correlation functions of the integrable spin-1/2 $XXX$ chain at arbitrary temperature/system size (no time for spin-1 or higher)

- Functional equations approach
- Factorized expressions for density matrix

Open problems: arbitrary spin-$S$, $XXZ$ and finite magnetic field – probably simple