The Pohlmeyer Reduction and q-Deformation of the $AdS_5 \times S^5$ Superstring

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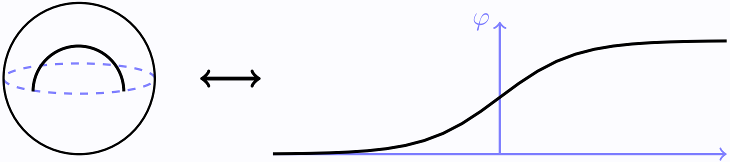
Part I based on unpublished work with Arkady Tseytlin

Part II based on arXiv:1206.0010 with Tim Hollowood and J. Luis Miramontes

Related papers include arXiv:0912.2958, arXiv:1104:2423 with Arkady Tseytlin and arXiv:1107.0628, arXiv:1112.4485 with Tim Hollowood and J. Luis Miramontes

Pohlmeyer reduction

Classical reformulation of superstring theory on $AdS_5 \times S^5$ and its subspaces



Key features

- Solves constraints $\longrightarrow$ describes physical degrees of freedom.
- Preserves classical integrability – equivalent integrable structure.
- Preserves 2-d Lorentz invariance and UV-finiteness.
- Standard kinetic terms for fermions – hidden 2-d SUSY.
- Special UV-finite massive integrable models

  — deserves study regardless of quantum equivalence.

Some history

The Pohlmeyer reduction relates the $O(n)$ sigma models (classical relativistic field theories) to integrable Hamiltonian systems.

- $O(3)$ sigma model classically related to sine-Gordon.
- $O(4)$ sigma model classically related to complex sine-Gordon.
- Discovery of the integrability of the classical $O(n)$ sigma model.
- Various developments, including use of Backlünd transformation to generate solutions and higher conserved charges.

Pohlmeyer '76
Luscher, Pohlmeyer '78
Pohlmeyer, Rehren '79
Eichenherr, Pohlmeyer '79

Pohlmeyer reduction – in string theory

- Interpret $O(n)$ sigma model as describing strings moving on S^{n-1}.
- Construction of classical string solutions in constant curvature backgrounds – de Sitter and anti de Sitter.

  Barbashov, Nesterenko '81
  de Vega, Sanchez '93
- Construction of classical string solutions representing semiclassical closed string states in AdS/CFT context.

  Hofman, Maldacena '06
  Dorey et al '06; Jevicki, Jin, Kalousios, Volovich '07
  BH, Iwashita, Tseytlin '09; Hollowood, Miramontes '09
  ...
- Study of Euclidean open-string world-surfaces – related to $\mathcal{N}=4$ super Yang-Mills scattering amplitudes at strong coupling.

  Alday, Maldacena '09; Alday, Gaiotto, Maldacena '09
  Dorn et al '09; Jevicki, Jin '09; Burrington, Gao '09
  ...
- Study of three- and four-point correlation functions.

  Janik, Wereszczyński '11; Kazama, Komatsu '11, '12
  Caetano, Toledo '12

Pohlmeyer reduction

Classical strings on S^2 (S^3) equivalent to (complex) sine-Gordon.

Pohlmeyer '76

$S^2 = \frac{SO(3)}{SO(2)}$ ↓ $S^3 = \frac{SO(4)}{SO(3)}$

Classical strings on the symmetric space $F/G, \dots$

related to the G/H gauged WZW theory plus integrable potential.

Bakas, Park, Shin '95; Grigoriev, Tseytlin '07

– symmetric space sine-Gordon models (SSSG)

Bakas, Fernandez-Pousa, Gallas, Hollowood, Miramontes, Park, Shin '94, '95 '96

Green-Schwarz action for Type IIB superstring theory on $AdS_5 \times S^5$ ↓ $\frac{PSU(2,2|4)}{USp(2,2) \times USp(4)}$

Metsaev, Tseytlin '98

Pohlmeyer-reduction of the $AdS_5 \times S^5$ superstring ...

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Pohlmeyer reduction of the $AdS_5 \times S^5$ superstring

Pohlmeyer-reduced $AdS_5 \times S^5$ superstring:

Gauged WZW model for

$$\frac{USp(2,2)}{SU(2)^2} \times \frac{USp(4)}{SU(2)^2}$$

plus integrable potential ... mixed through fermions

Grigoriev, Tseytlin '07; Mikhailov, Schäfer-Nameki '07

Key points:

- UV-finite — Roiban, Tseytlin '09
- One-loop partition function agrees with string theory – discrepancy in two-loop result, but suggestive relationship — BH, Iwashita, Tseytlin '09; Iwashita '10; Iwashita, Roiban, Tseytlin '11
- Conjectured soliton S-matrix given by "strong coupling" limit of q-deformed light-cone gauge-fixed string S-matrix — Beisert, Koroteev '08; Beisert '10; BH, Tseytlin '09,'10,'11 ; BH, Hollowood, Miramontes '11
- Only suitable for classical string solutions non-trivial in both AdS_5 and S^5

An action for the Pohlmeyer reduction of strings on AdS_n

Aims & Motivation

Find an action formulation for the Pohlmeyer reduction of strings on AdS_n.

Early work includes Grigoriev, Tseytlin '08 and Miramontes '08, but no action was found

Explore connection with Pohlmeyer reduction of strings on $\text{AdS}_n \times S^1$.

Check fluctuation frequencies for simple classical solutions – compare with string theory.

Used in the study of open-string world-surfaces and three-/four-point correlation functions.

Alday, Maldacena '09; Alday, Gaiotto, Maldacena '09
Dorn et al '09; Jevicki, Jin '09; Burrington, Gao '09
Janik, Wereszczyński '11; Kazama, Komatsu '11, '12; Caetano, Toledo '12

Construction of action and inclusion of fermions may eventually allow these computations to be generalized/extended.

Start with a review of the coordinate-based Pohlmeyer-reduction for strings on AdS_n.

Barbashov, Nesterenko '81; de Vega, Sanchez '93
Jevicki, Jin, Kalousios, Volovich '07
Dorn et al '09

Review of Pohlmeyer reduction of strings on AdS_n

Consider AdS_n embedded into $\mathbb{R}^{n-1,2}$. Embedding coordinates: Y_μ

$\mu = -1, 0, 1, \ldots, n-1$

World-sheet string action:

$$\mathcal{S} = \frac{1}{4\pi\alpha'} \int d^2x \big[\sqrt{-h} h^{\alpha\beta} \partial_\alpha Y \cdot \partial_\beta Y + \Lambda(Y \cdot Y + 1)\big]$$

Variational equations:

$$Y \cdot Y = -1 \ , \qquad \partial_\pm Y \cdot \partial_\pm Y = 0$$

$$\partial_+ \partial_- Y - (\partial_+ Y \cdot \partial_- Y) Y = 0$$

μ, ν contracted with $\mathrm{diag}(-1, -1, 1, 1, \ldots, 1)$.

Consider the vectors

$$Y_\mu \ , \qquad \partial_+ Y_\mu \ , \qquad \partial_- Y_\mu$$

They span a vector space – inner product restricted to this space has signature $(-,-,+)$

assuming time-like world-sheet

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Can find $n-2$ vectors N_i that satisfy

$$N_i \cdot Y = N_i \cdot \partial_+ Y = N_i \cdot \partial_- Y = 0 \ , \qquad N_i \cdot N_j = \delta_{ij}$$

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$$N_i \cdot Y = N_i \cdot \partial_+ Y = N_i \cdot \partial_- Y = 0 \ , \qquad N_i \cdot N_j = \delta_{ij}$$

Define the fields ϕ, u_i, v_i, $B_{\pm\, ij}$

$$\partial_+ Y \cdot \partial_- Y = -\frac{1}{2} e^{2\phi} \ , \quad N_i \cdot \partial_+^2 Y = \frac{1}{2} u_i \ , \quad N_i \cdot \partial_-^2 Y = -\frac{1}{2} v_i$$

$$B_{ij} = dN_i \cdot N_j \ (= -dN_j \cdot N_i)$$

Given the string equations of motion these new fields satisfy the following equations

$$\partial_+\partial_-\phi + \frac{1}{4}(e^{2\phi} - u_i v_i e^{-2\phi}) = 0$$

$$\partial_- u_i = B_{-\,ij} u_j \ , \qquad \partial_+ v_i = B_{+\,ij} v_j$$

$$\begin{aligned} F_{-+\,ij} \equiv & \partial_- B_{+\,ij} - \partial_+ B_{-\,ij} - B_{-\,ik} B_{+\,kj} + B_{+\,ik} B_{-\,kj} \\ & = \frac{1}{2} e^{-2\phi} (u_i v_j - u_j v_i) \end{aligned}$$

Some properties:

- $SO(n-2)$ gauge symmetry (B is gauge field, u, v are vectors)
- $n = 2$ gives the Liouville equation – conformal theory

$$\partial_+ \partial_- \phi + \frac{1}{4} e^{2\phi} = 0$$

- Invariant under local conformal reparametrizations

$$(\partial_+, B_+) \to \Lambda^{\uparrow} (\partial_+, B_+) \ , \qquad (\partial_-, B_-) \to \Lambda^{\downarrow} (\partial_-, B_-) \ , \qquad \Lambda^{\uparrow} = \Lambda^{\uparrow}(x^+)$$
$$e^{2\phi} \to \Lambda^{\uparrow} \Lambda^{\downarrow} e^{2\phi} \ , \qquad u \to \Lambda^{\uparrow 2} u \ , \qquad v \to \Lambda^{\downarrow 2} v \ , \qquad \Lambda^{\downarrow} = \Lambda^{\downarrow}(x^-)$$

Action for the Pohlmeyer reduced string on AdS space

Plan:

- To regulate the reduction procedure we introduce extra S^1 with coordinate θ – gauge-fixed equal to $\mu\tau$
- Pohlmeyer reduction of strings on $AdS_n \times S^1$ is well-known and has action formulation
- Modify by introducing asymmetric gauging (depending on μ) of WZW model
- Take $\mu \to 0$ to be left with non-trivial string on AdS_n

Write AdS_n as symmetric coset space

$$\mathrm{AdS}_n \cong \frac{F}{G} = \frac{SO(n-1,2)}{SO(n-1,1)}$$

$$\mathfrak{f} = \mathfrak{so}(n-1,2)\ , \qquad \mathfrak{g} = \mathfrak{so}(n-1,1)\ , \qquad \mathfrak{f} = \mathfrak{g} \oplus \mathfrak{p}$$

Introduce group-valued field $f \in F$ and left-invariant MC one-form

$$\mathcal{J} = f^{-1} df = \mathcal{P} + \mathcal{A} \in \mathfrak{p} \oplus \mathfrak{g}$$

World-sheet action for strings on $\mathrm{AdS}_n \times S^1$

$$\mathcal{S} = \frac{1}{4\pi\alpha'} \int d^2x\, \sqrt{-h} h^{ab} \Big[\frac{1}{2}\,\mathrm{Tr}(\mathcal{P}_a \mathcal{P}_b) + \partial_a\theta \partial_b\theta\Big]$$

θ coordinate on S^1

Fix conformal gauge ($\sqrt{-h}h = \eta$) and also $\theta = \mu\tau$

Equations of motion:

$$\mathcal{D}_+\mathcal{P}_- + \mathcal{D}_-\mathcal{P}_+ = 0$$

Virasoro constraints:

$$-\frac{1}{2}\,\mathrm{Tr}(\mathcal{P}_\pm\mathcal{P}_\pm) = \mu^2$$

Flatness of MC one-form:

$$\mathcal{D}_+\mathcal{P}_- - \mathcal{D}_-\mathcal{P}_+ = 0$$

$$d\mathcal{A} + \mathcal{A}\wedge\mathcal{A} + \mathcal{P}\wedge\mathcal{P} = 0$$

Want to solve the Virasoro constraints:

$$-\frac{1}{2}\,\mathrm{Tr}(\mathcal{P}_\pm \mathcal{P}_\pm) = \mu^2$$

Standard route:

Maximal abelian subalgebra of $\mathfrak{p}$ is one-dimensional

Pick a particular candidate generated by element T, such that

$$\mathrm{Tr}(T^2) = -2$$

Can then use $SO(n-1,1)$ gauge symmetry to write

$$\mathcal{P}_+ = \mu\, T\ , \qquad \mathcal{P}_- = \mu\, g^{-1} T g\ , \qquad g \in G = SO(n-1,1)$$

Notation: denote subalgebra of $\mathfrak{g}$ that commutes with T as $\mathfrak{h}$ and its orthogonal complement in $\mathfrak{g}$ as $\mathfrak{m}$

$$[T, \mathfrak{h}] = 0\ , \qquad \mathfrak{h} = \mathfrak{so}(n-1)\ , \qquad \mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$$

Want to solve the Virasoro constraints:

$$-\frac{1}{2}\,\mathrm{Tr}(\mathcal{P}_\pm \mathcal{P}_\pm) = \mu^2$$

New route:

Still need T, but also consider element of $\mathfrak{m}$, R, such that

$$\mathrm{Tr}(R^2) = 2\ , \qquad [T,\, R] = S \in \mathfrak{p}\ , \qquad \mathrm{Tr}(S^2) = 2$$

Notation: denote subalgebra of $\mathfrak{h}$ that commutes with R as $\mathfrak{k}$
and its orthogonal complement in $\mathfrak{h}$ as $\mathfrak{l}$

$$[R,\, \mathfrak{k}] = 0\ , \qquad \mathfrak{k} = \mathfrak{so}(n-2)\ , \qquad \mathfrak{h} = \mathfrak{k} \oplus \mathfrak{l}$$

Want to solve the Virasoro constraints:

$$-\frac{1}{2}\,\mathrm{Tr}(\mathcal{P}_\pm \mathcal{P}_\pm) = \mu^2$$

New route:

$$\mathfrak{f} = \mathfrak{so}(n-1,2) = \mathfrak{g} \oplus \mathfrak{p} \dashrightarrow T \in \mathfrak{p} : [T, \mathfrak{h}] = 0$$

$$\downarrow$$

$$\mathfrak{g} = \mathfrak{so}(n-1,1) = \mathfrak{h} \oplus \mathfrak{m} \dashrightarrow R \in \mathfrak{m} : [R, \mathfrak{k}] = 0$$

$$\downarrow$$

$$\mathfrak{h} = \mathfrak{so}(n-1) = \mathfrak{k} \oplus \mathfrak{l}$$

$$\downarrow$$

$$\mathfrak{k} = \mathfrak{so}(n-2)$$

Want to solve the Virasoro constraints:

$$-\frac{1}{2}\,\mathrm{Tr}(\mathcal{P}_\pm\mathcal{P}_\pm) = \mu^2$$

New route:

Introduce the algebra automorphisms

$$\epsilon_\pm(\mathfrak{J}) = \mu^{\mp R}\mathfrak{J}\mu^{\pm R}\ , \qquad \mu^{aR} = e^{aR\log\mu}$$

Solve the Virasoro constraints

$$\mathcal{P}_+ = \mu\,\epsilon_-(T)\ , \qquad \mathcal{P}_- = \mu\, g^{-1}\epsilon_+(T)g$$

Key point: Unlike before this parametrization has non-trivial limit

$$\lim_{\mu\to 0}\mathcal{P}_+ = T_-\ , \qquad \lim_{\mu\to 0}\mathcal{P}_- = g^{-1}T_+g$$

$$T_\pm = \frac{1}{2}(T \mp S)\ , \qquad \mathrm{Tr}(T_\pm^2) = 0$$

Want to solve the equations of motion:

$$\mathcal{D}_\pm \mathcal{P}_\mp = 0$$

For now, keep μ finite

Equations of motion are solved by introducing two new fields $A_\pm$

$$\mathcal{A}_+ = g^{-1}\partial_+ g + g^{-1}\epsilon_+(A_+)g \ , \qquad \mathcal{A}_- = \epsilon_-(A_-) \qquad A_\pm \in \mathfrak{h}$$

‘

Substituting this, along with $\mathcal{P}_\pm$ in terms of g, into the flatness equation projected onto $\mathfrak{g}$, gives us . . .

0. Equation of motion for the reduced theory:

$$\partial_-(g^{-1}\partial_+ g + g^{-1}\epsilon_+(A_+)g) - \partial_+\epsilon_-(A_-) + [\epsilon_-(A_-), g^{-1}\partial_+ g + g^{-1}\epsilon_+(A_+)g] + \mu^2[g^{-1}\epsilon_+(T)g, \epsilon_-(T)] = 0$$

1. Take the $\mu \to 0$ limit

2. Then gauge-fix to find a Lagrangian set of equations

Step 1. Take the $\mu \to 0$ limit

Regarding the potential term, we have already seen that

$$\lim_{\mu\to 0} \mu\, \epsilon_\pm(T) = T_\pm$$

For the terms involving the gauge field, we decompose

$$A_\pm = B_\pm + \tilde{C}_\pm \ , \qquad B_\pm \in \mathfrak{k}, \quad \tilde{C}_\pm \in \mathfrak{l}$$

such that

$$\epsilon_\pm(B_\pm) = B_\pm \ , \qquad \lim_{\mu\to 0} \epsilon_\pm(\tilde{C}_\pm) = C_\pm \in \mathfrak{l}_\pm$$

assuming $\tilde{C}_\pm$ scale like μ

Step 1. Take the $\mu \to 0$ limit

Generators of the spaces $\mathfrak{l}_\pm$ are linear combinations of those in $\mathfrak{l}$ and those in $\mathfrak{m}$ (not including R)

Key features:

- $[T_\pm, \mathfrak{l}_\pm] = 0$
- $\mathfrak{l}_\pm$ are abelian ($\mathbb{R}^{n-2}$)
- Transform as vectors of $SO(n-2)$ under adjoint action of K

Therefore $\mathfrak{k} \oplus \mathfrak{l}_\pm$ contain generators of two copies of the Euclidean group E_{n-2}, with common $SO(n-2)$ subgroup.

Contraction of the algebra $\mathfrak{h} = \mathfrak{so}(n-1)$

Step 2. Partially gauge-fix

Equation of motion has an $E_{n-2\,R} \times E_{n-2\,L}$ gauge symmetry

- i. Gauge-fix $C_\pm = 0$, left with $SO(n-2)_R \times SO(n-2)_L$ gauge symmetry
- ii. Partially gauge-fix to leave a single $SO(n-2)$ gauge symmetry
- iii. Resulting equations come from the following action

$$\mathcal{S} = \frac{\ell}{8\pi} \mathrm{Tr} \Big[\frac{1}{2} \int d^2x \;\; g^{-1}\partial_+ g \; g^{-1}\partial_- g \;\; - \frac{1}{3} \int d^3x \;\; \epsilon^{mnl} \; g^{-1}\partial_m g \; g^{-1}\partial_n g \; g^{-1}\partial_l g$$
$$+ \int d^2x \;\; \left(B_+\partial_- g g^{-1} - B_- g^{-1}\partial_+ g - g^{-1}B_+ g B_- + B_+ B_-\right)$$
$$+ \int d^2x \;\; g^{-1} T_+ g T_- \Big] \, .$$

Gauged WZW for $\frac{G}{K} = \frac{SO(n-1,1)}{SO(n-2)}$ plus potential

Claim: this action gives the required equations of motion

If we parametrize

$$g = k^{-1} e^{\xi_+} e^{-2\phi R} e^{\xi_-} \ , \qquad k \in K \ , \quad \xi_\pm \in \mathfrak{l}_\pm$$

and redefine $B_+ \to k^{-1} B_+ k + k^{-1} \partial_+ k$

then the action above can be rewritten as:

$$\begin{aligned} \mathcal{S} = \ & \frac{\ell}{2\pi} \int d^2x \, \left[\partial_+\phi\partial_-\phi - \frac{1}{4} e^{2\phi}\right] + \frac{\ell}{8\pi} \int d^2x \ e^{2\phi} \, \mathrm{Tr} \left[D_-\xi_- D_+\xi_+\right] \\ & - \frac{\ell}{8\pi} \mathrm{Tr} \Big[\frac{1}{2} \int d^2x \ \ k^{-1}\partial_+ k k^{-1}\partial_- k \ - \frac{1}{3} \int d^3x \ \ \epsilon^{mnl} \ k^{-1}\partial_m k \ k^{-1}\partial_n k \ k^{-1}\partial_l k \\ & \qquad + \int d^2x \ \ (B_+\partial_- k k^{-1} - B_- k^{-1}\partial_+ k - k^{-1} B_+ k B_- + B_+ B_-) \Big] \ . \end{aligned}$$

Liouville action for ϕ, gauged WZW for $\frac{K}{K}$, mixed through terms involving $\xi_\pm$.

Equations of motion:

$$\phi \;:\quad \partial_+\partial_-\phi + {}^1\!/\!_4 e^{2\phi}(1 - \mathrm{Tr}[D_-\xi_- D_+\xi_+]) = 0\ , \qquad \xi_\pm \;:\quad D_\pm(e^{2\phi}D_\mp\xi_\mp) = 0$$

$$B_+ \;:\quad B_- - kB_-k^{-1} + \partial_-kk^{-1} = e^{2\phi}[\xi_+,\, D_-\xi_-]\Big|_{\mathfrak{k}}$$

$$B_- \;:\quad B_+ - k^{-1}B_+k - k^{-1}\partial_+k = e^{2\phi}[\xi_-,\, D_+\xi_+]\Big|_{\mathfrak{k}}$$

$$k \;:\quad D_-(k^{-1}B_+k + k^{-1}\partial_+k) = \partial_+B_- \qquad \text{or} \qquad D_+(kB_-k^{-1} - \partial_-kk^{-1}) = \partial_-B_+$$

Equations of motion:

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$$B_- \ : \quad B_+ - k^{-1}B_+k - k^{-1}\partial_+k = e^{2\phi}[\xi_-, \, D_+\xi_+]\Big|_{\mathfrak{k}}$$

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Defining $\qquad u = e^{2\phi}D_+\xi_+ \ , \qquad v = e^{2\phi}D_-\xi_-$

and eliminating k, these equations are equivalent to

Equations of motion:

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Defining $\qquad u = e^{2\phi}D_+\xi_+ \;, \qquad v = e^{2\phi}D_-\xi_-$

and eliminating k, these equations are equivalent to

$$\partial_+\partial_-\phi + {}^1\!/\!_4 e^{2\phi} + {}^1\!/\!_4 e^{-2\phi}\,\mathrm{Tr}[u\,v] = 0 \;,$$
$$D_+v = D_-u = 0 \;, \qquad F_{-+} = e^{-2\phi}[u, \, v]\Big|_{\mathfrak{k}} \;.$$

which (expanding in components) are the same as those found from the coordinate-based reduction earlier.

Conclusions

- Have found action formulation for the Pohlmeyer-reduced string on AdS_n
- Connected to the Pohlmeyer reduction of strings on $AdS_n \times S^1$ through $\mu \to 0$ limit

- This limiting procedure can be extended to full superstring
- Gives action for Pohlmeyer reduced $AdS_5 \times S^5$ superstring for classical solutions living only on AdS_5 ☺
- Fluctuation frequencies around classical solution corresponding to large spin limit of GKP string agree with those from string theory (all $8+8$) – possible matching of one-loop partition function?

- Open questions
  - ∗ In fluctuation computation, extra massless modes, coming from presence of $\xi_\pm$ in action rather than u and v (need to be removed by hand)
  - ∗ Chose time-like world-sheet – leads to action with ghost-like kinetic terms – doesn't affect frequencies – why are they there or how is this resolved?
  - ∗ For space-like world-sheet we Wick rotate $K = SO(n-2) \to SO(1, n-1)$ – correct signs for all kinetic terms . . .
  - ∗ Applications to more complicated string solutions – open strings etc.

The q-deformed light-cone gauge-fixed superstring

Aims & Motivation

Study the q-deformation of the light-cone gauge-fixed superstring R-matrix – solve crossing equation to find the phase and the S-matrix

Beisert, Koroteev '08; Beisert '10
BH, Hollowood, Miramontes 11, '12

Taking q – the deformation parameter – to be a phase ($\exp({}^{i\pi}\!/_{k})$) in the strong-coupling limit we find the conjectured R-matrix underlying the scattering of solitons in the PR theory.

BH, Tseytlin '11
BH, Hollowood, Miramontes '11

Important open questions regarding unitarity and the matching with the perturbative result (does not satisfy YBE)

BH, Tseytlin '10, '11

Provides "quantum" connection between the Pohlmeyer-reduced and light-cone gauge-fixed superstrings ☺

Quantum analogue to classical interpolating Poisson structure – possible connections to the recent generalization of the Faddeev-Reshetikin procedure to the superstring

Delduc, Magro, Vicedo '12
see also – Marc's talk later today

TBA for interpolating theory - deformation parameter q may provide regularization

Arutyunov, de Leeuw, van Tongeren '12
see also – Stijn's talk later today

Q-deformation of the light-cone gauge-fixed algebra

Start from the algebra $\mathfrak{psu}(2|2)^{\oplus 2} \ltimes \mathbb{R}^3$ and consider just one factor

Two $\mathfrak{su}(2)$ algebras, with generators: $\mathfrak{R}^a_b$ $\mathfrak{L}^\alpha_\beta$

Fermionic generators, charged under $\mathfrak{su}(2)^{\oplus 2}$: $\mathfrak{Q}^a_\beta$ $\mathfrak{S}^\alpha_b$

Central extensions: $\mathfrak{P}$ $\mathfrak{K}$ $\mathfrak{C}$

$$\{\mathfrak{Q}, \mathfrak{Q}\} \sim \mathfrak{P} \,, \qquad \{\mathfrak{S}, \mathfrak{S}\} \sim \mathfrak{K} \,, \qquad \{\mathfrak{Q}, \mathfrak{S}\} \sim \mathfrak{R} + \mathfrak{L} + \mathfrak{C} \,.$$

To quantum deform go to Chevalley basis plus Serre relations, and modify the defining relations, through introduction of a new parameter, q.

Beisert, Koroteev '08
see also – Alessandro's talk on Wednesday

Q-deformation of the coproduct

Interested in the scattering of two particles
$\longrightarrow$ Need the R-matrix and action of symmetry on two-particle states
$\longrightarrow$ Introduce the coproduct

$$\Delta(\mathfrak{J}) = \mathfrak{J} \otimes \star_1 + \star_2 \otimes \mathfrak{J} \qquad \mathfrak{J} \in \text{Chevalley basis}$$

For consistency of coproduct with the product (Lie-bracket) in the quantum-deformed theory $\star_1$ and $\star_2$ are non-trivial multiplicative combinations of

$$I,\ q^{\mathfrak{R}^1_1},\ q^{\mathfrak{L}^1_1},\ \mathfrak{V} = q^{\mathfrak{C}} \text{ and } \mathfrak{U}$$

$\mathfrak{U}$ is braiding factor that is introduced in the light-cone gauge-fixed string theory construction – related to the world-sheet momentum

For the central elements to have a trivial coproduct

$$q^C = V \ , \qquad P = g(1 - U^2 V^2) \ , \qquad K = g(V^{-2} - U^{-2}) \ ,$$

Non-gothic denotes the eigenvalues of the central elements

Couplings

The theory we are discussing therefore has two couplings

$$g \text{ and } q$$

We will consider q to be a phase

$$q = \exp\left(\frac{i\pi}{k}\right)$$

In the $g \to \infty$ limit k then becomes the coupling of the PR theory.

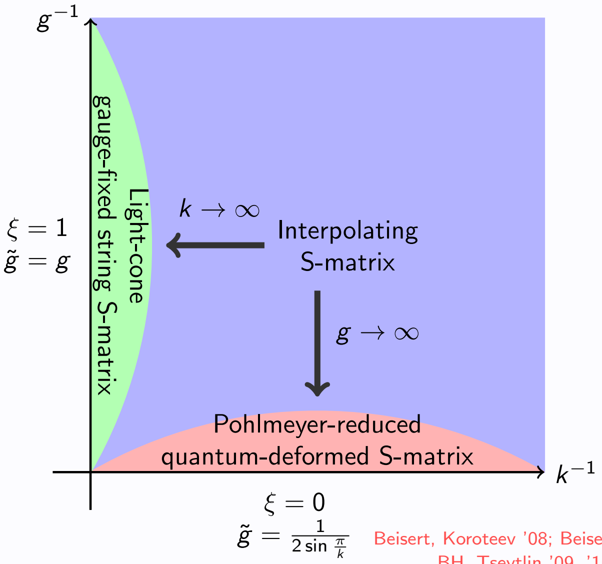
In the $k \to \infty$ limit g is proportional to the string tension $\sqrt{\lambda}$.

Also useful to consider the alternative pair of couplings

$$\tilde{g} = \frac{g}{\sqrt{1 + 4g^2 \sin^2 \frac{\pi}{k}}} \quad \text{and} \quad \xi = 2\tilde{g} \sin \frac{\pi}{k}$$

Beisert, Galleas, Matsumoto '11

Interpolating S-matrix



Beisert, Koroteev '08; Beisert '10
BH, Tseytlin '09, '10, '11
BH, Hollowood, Miramontes '11

Single particle states

Single particle states are modules corresponding to the (4×4)-dimensional representation of $\mathcal{U}_q(\mathfrak{psu}(2|2)^{\oplus 2} \ltimes \mathbb{R}^3)$.

$$\left|\Phi^{A\dot{A}}\right\rangle \qquad A = (a, \alpha) \quad \dot{A} = (\dot{a}, \dot{\alpha})$$

The quantum deformation modifies the shortening condition – constraint coming from the closure of the algebra acting on the representation

$$C^2 - PK = \frac{1}{4} \quad \longrightarrow \quad \left[C\right]_q^2 - PK = \left[\frac{1}{2}\right]_q^2$$

where $[x]_q = (q^x - q^{-x})/(q - q^{-1})$ is the usual quantum number

Single particle states

C, P and K can be written in terms of U and V, which satisfy the constraint equation

$$\big[C\big]_q^2 - PK = \big[\tfrac{1}{2}\big]_q^2$$
$$\Rightarrow \xi\,(V^2 + V^{-2}) - \xi^{-1}(U^2 + U^{-2}) = (\xi - \xi^{-1})(q + q^{-1})$$

Introduce the usual $x^{\pm}$ variables

$$U^2 = q^{-1}\frac{x^+ + \xi}{x^- + \xi} = q\,\frac{\frac{1}{x^-} + \xi}{\frac{1}{x^+} + \xi}\,, \qquad V^2 = q^{-1}\frac{\xi x^+ + 1}{\xi x^- + 1} = q\,\frac{\frac{\xi}{x^-} + 1}{\frac{\xi}{x^+} + 1}\,,$$

whose constraint equation is ...

$$q^{-1}\Big(x^+ + \frac{1}{x^+}\Big) - q\Big(x^- + \frac{1}{x^-}\Big) = (q - q^{-1})\Big(\xi + \frac{1}{\xi}\Big)\,,$$

In $q \to 1$ limit right-hand-side becomes $\frac{i}{g}$.

and label states with $x_i^{\pm} \leftrightarrow x_i$

The q-deformed Ř-matrix

The $\check{R}$-matrix:

$$\check{R}(x_1^+, x_1^-; x_2^+, x_2^-) : \ \mathbf{V}(x_1) \otimes \mathbf{V}(x_2) \longrightarrow \mathbf{V}(x_2) \otimes \mathbf{V}(x_1) \ ,$$

$\mathbf{V}(x_i)$ is the (4×4)-dimensional module spanned by the basis $\left\{|\Phi^{A\dot{A}}\rangle\right\}$

is completely fixed by symmetry, up to a phase

$$\check{R}\Delta(\mathfrak{J}) = \Delta(\mathfrak{J})\check{R}$$

Beisert, Koroteev '08

Infinite dimensional q-deformed affine symmetry fixes bound state $\check{R}$-matrices also up to phase.

Beisert, Galleas, Matsumoto '11
de Leeuw, Matsumoto, Regelskis '11

The S-matrices are given by these $\check{R}$-matrices with the phases fixed by unitarity, crossing and fusion.

Q-deformed S-matrix – the phase

Phase of light-cone gauge-fixed S-matrix

including . . . Arutyunov, Frolov, Staudacher '04
Beisert, Staudacher '05
Beisert, Tseytlin '05
Janik '06
Hernandez, Lopez '06
Arutyunov, Frolov '06
Freyhult, Kristjansen '06
Beisert, Hernandez, Lopez '06
Beisert '06
Beisert, Eden, Staudacher '06
Arutyunov, Frolov, Zamaklar '06
Kostov, Serban, Volin '07
Dorey, Hofman, Maldacena '07
Gromov, Vieira '07
Arutyunov, Frolov '09
Volin '09
Review article – Vieira, Volin '10

Long story!

Q-deformed S-matrix phase construction follows a similar logic.

Solution of crossing equation

Following light-cone gauge-fixed string theory ...

$$\sigma(x_1^\pm, x_2^\pm) = \exp i\big[\chi(x_1^+, x_2^+) - \chi(x_1^-, x_2^+) - \chi(x_1^+, x_2^-) + \chi(x_1^-, x_2^-)\big]$$

Solve the associated Riemann-Hilbert problem to find:

$$\chi(x_1, x_2) = i \oint_{|z|=1} \frac{dz}{2\pi i} \frac{1}{z - x_1} \oint_{|z'|=1} \frac{dz'}{2\pi i} \frac{1}{z' - x_2} \log \frac{\Gamma_{q^2}(1 + iu(z) - iu(z'))}{\Gamma_{q^2}(1 - iu(z) + iu(z'))}$$

$$\Gamma_{q^2}(1+x) = \frac{1 - q^{2x}}{1 - q^2} \Gamma_{q^2}(x)$$

BH, Hollowood, Miramontes '11

String limit: $q \to 1 \Rightarrow \Gamma_{q^2}(x) \to \Gamma(x)$ and as expected we get the light-cone gauge-fixed string phase ☺

PR limit: $g \to \infty \to$ can show the phase reduces to the correct relativistic expression ☺

Something is a bit strange

Elementary particles are in the fundamental representation

$$\langle 0,0\rangle^2 = \big((1,0)\oplus(0,1)\big)^2$$

In the string limit $k \to \infty$ bound states transform in the representation

$$\langle 1,0\rangle^2 = \big((2,0)\oplus(1,1)\oplus(0,0)\big)^2$$

In the PR limit $g \to \infty$ bound states transform in the representation

$$\langle 0,1\rangle^2 = \big((0,2)\oplus(1,1)\oplus(0,0)\big)^2$$

Something is a bit strange

In both the light-cone gauge-fixed and Pohlmeyer-reduced theories the bound states should form in higher representations of the $\mathfrak{su}(2)^2$ subalgebra that comes from the sphere sector of the theory.

In the two limits this is innocuous as we can just associate a different $\mathfrak{su}(2)^2$ algebra to the $\mathfrak{su}(4)$ of the sphere.

Algebraically this is fine as the associated modules are isomorphic (including fermion number).

But what happens as we move between the two theories?

The dispersion relation

The constraint equation for $x^{\pm}$

$$q^{-1}\Big(x^+ + \frac{1}{x^+} - \xi - \frac{1}{\xi}\Big) = q\Big(x^- + \frac{1}{x^-} - \xi - \frac{1}{\xi}\Big) \ ,$$

is consistent with usual light-cone gauge-fixed string reality condition

$$(x^+)^* = x^- \qquad \Rightarrow \qquad UU^* = VV^* = 1 \ .$$

$$U = e^{\frac{ip}{2g}} \qquad V = e^{\frac{i\xi E}{2g}}$$

$$\cos(\xi E/g) - \xi^2 \cos(p/g) = (1-\xi^2)\cos(\pi/k)$$

$k \to \infty$ $\qquad$ $g \to \infty$

$$E^2 = \tfrac{1}{4} + 4g^2 \sin^2 p/2g \ , \qquad E^2 - p^2 = \left(2\cos \pi/2k\right)^{-2}$$

The dispersion relation

The constraint equation for $x^\pm$

$$q^{-a}\Big(x^+ + \frac{1}{x^+} - \xi - \frac{1}{\xi}\Big) = q^{a}\Big(x^- + \frac{1}{x^-} - \xi - \frac{1}{\xi}\Big) \ ,$$

is consistent with usual light-cone gauge-fixed string reality condition

$$(x^+)^* = x^- \qquad \Rightarrow \qquad UU^* = VV^* = 1 \ .$$

$$U = e^{\frac{ip}{2g}} \qquad V = e^{\frac{i\xi E}{2g}}$$

$$\cos({}^{\xi E}\!/_{g}) - \xi^2 \cos({}^{p}\!/_{g}) = (1-\xi^2)\cos({}^{a\pi}\!/_{k})$$

The dispersion relation

The constraint equation for $x^{\pm}$

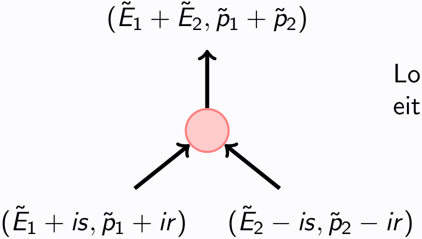
$$q^{-a}\left(x^{+}+\frac{1}{x^{+}}-\xi-\frac{1}{\xi}\right)=q^{a}\left(x^{-}+\frac{1}{x^{-}}-\xi-\frac{1}{\xi}\right) ,$$

is consistent with usual light-cone gauge-fixed string reality condition

$$(x^{+})^{*}=x^{-} \qquad \Rightarrow \qquad UU^{*}=VV^{*}=1 \; .$$

$$U=e^{\frac{ip}{2g}} \qquad V=e^{\frac{i\xi E}{2g}}$$

$$\cos({}^{\xi E}\!/\!_{g})-\xi^{2}\cos({}^{p}\!/\!_{g})=(1-\xi^{2})\cos({}^{a\pi}\!/\!_{k})$$



Look for solutions when either $x_1^{+}=x_2^{-}$ or $x_1^{-}=x_2^{+}$

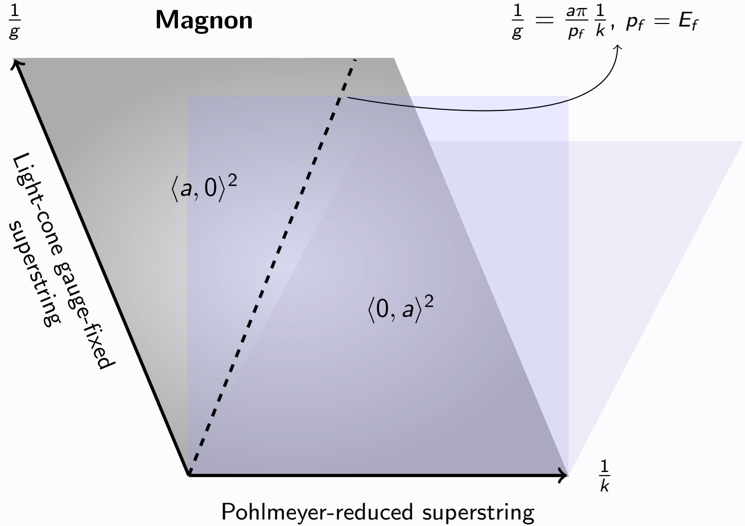
Two branches of solutions

- $|p| < \xi E \ , \qquad |p| < \frac{a\pi g}{k} \ , \qquad q^{-2iu} \in \mathbb{R} < 0 \ , \qquad \langle 0, a \rangle^2 \ ,$
- $|p| > \xi E \ , \qquad \frac{a\pi g}{k} < |p| \ , \qquad q^{-2iu} \in \mathbb{R} > 0 \ , \qquad \langle a, 0 \rangle^2 \ .$

In a relativistic theory, Lorentz transformations can be used to boost the momentum of a state. The intrinsic properties of the state, however, do not change. Here this intuition must be abandoned in the non-relativistic interpolating theory: the nature of the state depends on its momentum.

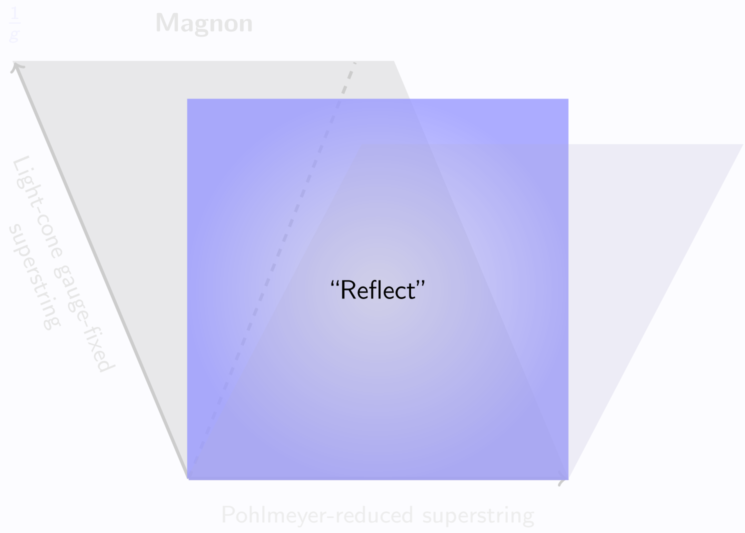
Special point at $p = \xi E = \frac{a\pi g}{k}$ for which the bound states become marginally unstable.

Magnon



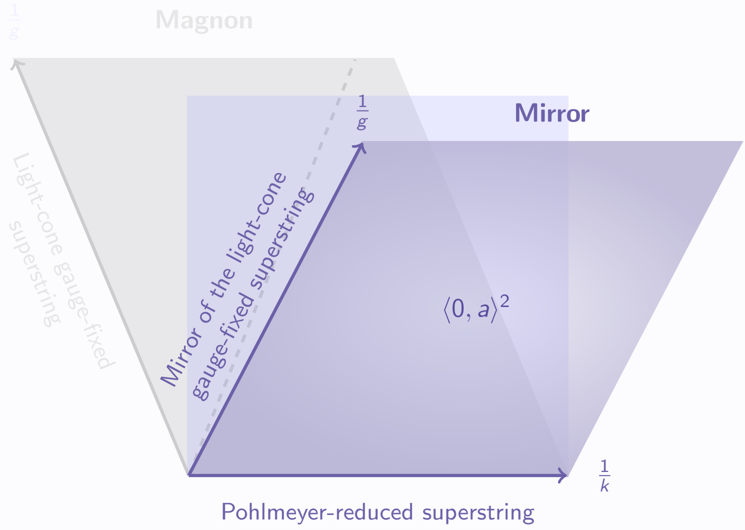
Picture for fixed momentum $p = p_f$

BH, Hollowood, Miramontes '12



Picture for fixed momentum $p = p_f$

BH, Hollowood, Miramontes '12



Picture for fixed momentum $p = p_f$

BH, Hollowood, Miramontes '12

Conclusions

- Q-deformed superstring S-matrix and its mirror have been constructed.
- Quantum connection between light-cone gauge-fixed and Pohlmeyer-reduced superstrings ☺.

- Open questions
  - PR S-matrix - relation between perturbative and q-deformed S-matrices.
  - Matrix unitarity of q-deformed S-matrix.
  - Relation of S-matrix story to semi-classical partition function.
  - Other limits of interpolating S-matrix – double scalings of g and k.
  - $g \to \infty$ limit of the underlying quantum affine symmetry.
  - Physical meaning of interpolating theory?

Thank you!