Yangian symmetry of scattering amplitudes in planar $\mathcal{N} = 4$ Super Yang-Mills

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with Simon Caron-Huot, 1112.1060 and work in progress.

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Plan of the talk

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- The S-matrix from the symmetry
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- Summary and outlook
The symmetry of the S-matrix in planar $\mathcal{N} = 4$ SYM

- All the on-shell states in $\mathcal{N} = 4$ SYM can be combined into an on-shell superfield,

\[
\Phi = G^+ + \eta^A \Gamma_A + \frac{1}{2!} \eta^A \eta^B S_{AB} + \frac{1}{3!} \varepsilon_{ABCD} \eta^A \eta^B \eta^C \Gamma^D + \frac{1}{4!} \varepsilon_{ABCD} \eta^A \eta^B \eta^C \eta^D G^- ,
\]

which depends on the Grassmann variable $\eta^A$, and a null momenta $p_{\alpha \dot{\alpha}} = \lambda_{\alpha} \bar{\lambda}_{\dot{\alpha}}$.

- All color-ordered amplitudes are packaged into a superamplitude $A(\{\lambda_i, \bar{\lambda}_i, \eta_i\})$; it can be classified according to the Grassmann degree $4k + 8$,

\[
A_n = A_{n, \text{MHV}} + A_{n, \text{NMHV}} + \cdots + A_{n, \text{MHV}} = \frac{\delta^4(\sum_i \lambda_i \bar{\lambda}_i) \delta^0|8(\sum_i \lambda_i \eta_i)}{\langle 12 \rangle \langle 23 \rangle \cdots \langle n1 \rangle} \sum_{k=0}^{n-3} A_{n,k}.
\]

where we strip off the MHV tree prefactor; $A_{n,k}$ denotes the $N^k$ MHV amplitude.

- $\mathcal{N} = 4$ SYM is a superconformal field theory. By introducing a deformation of the free algebra, the tree-level S-matrix is invariant under this $\mathfrak{psu}(2,2|4)$ symmetry:

\{
q^\alpha_A, \bar{q}^A_\dot{\alpha}, p_{\alpha \dot{\alpha}}, m_{\alpha \beta}, \bar{m}_{\dot{\alpha} \dot{\beta}}, s^A_\alpha, \bar{s}^A_\dot{\alpha}, e^{\alpha \dot{\alpha}}, d, r^A_B\} [\text{Bargheer Beisert Galleas Loebbert McLoughlin 2009}].
The symmetry of the S-matrix in planar $\mathcal{N} = 4$ SYM

\begin{align*}
x^\alpha_{i} \bar{\alpha}_{i} - x^\alpha_{i-1} \bar{\alpha}_{i-1} &= \lambda^\alpha_i \bar{\lambda}_i, \\
\theta^A_i - \theta^A_{i-1} &= \lambda^\alpha_i \eta^A_i.
\end{align*}
Yangian symmetry of the S-matrix in planar $\mathcal{N} = 4$ SYM

- In the planar limit, a dual conformal symmetry has been observed at both weak [Drummond Henn Smirnov Sokatchev 2006] and strong couplings [Alday Maldacena 2007]. The symmetry has been generalized to a dual superconformal symmetry [Drummond Henn Korchemsky Sokatchev 2008]. The tree-level S-matrix is invariant under the dual $\text{psu}(2,2|4)$ symmetry.

- The four-gluon amplitude has an all-loop, exponentiated form [Anastasiou Bern Dixon Kosower 2003],

  \[ A_4 = \exp[-\Gamma_{\text{cusp}} \log \frac{-s - i\epsilon}{\mu^2} \log \frac{-t}{\mu^2} + d(\log \frac{-s - i\epsilon}{\mu^2} + \log \frac{-t}{\mu^2}) + \text{const}]. \]

A general ansatz to remove all infrared and collinear divergences [Bern Dixon Smirnov 2005]:

\[ A_{n}^{\text{BDS}} = 1 + \sum_{\ell=1}^{\infty} g^{2\ell} A_{n}^{(\ell)}(\epsilon) = \exp \left[ \sum_{\ell=1}^{\infty} g^{2\ell} \left( \Gamma_{\text{cusp}}(\epsilon) A_{n,0}^{(1)}(\ell\epsilon) + C^{(\ell)} + E_{n}^{(\ell)}(\epsilon) \right) \right]. \]

- Loop amplitudes are not invariant under the dual conformal symmetry, but they satisfy an anomalous Ward identity [Drummond Henn Korchemsky Sokatchev 2007]. BDS ansatz is exact for $n = 4, 5$, since it is the only solution. In general, a finite remainder function is allowed, which depends on $3(n - 5)$ cross-ratios, e.g. $u_1 = \frac{x_{13}^2 x_{46}^2}{x_{14}^2 x_{36}^2}$ etc. for $n = 6$. 
The symmetry of the S-matrix in planar $\mathcal{N} = 4$ SYM


- The original superconformal symmetry of the amplitude are mapped to the dual symmetry of the Wilson loop by T-dualities \cite{Berkovits:2008ic, Beisert:2008tw}. Their closure is an infinite-dimensional Yangian symmetry, $\gamma[p\mathfrak{su}(2, 2|4)]$ \cite{Drummond:2009fd}.

- A generalized duality between the superamplitude and a supersymmetric Wilson loop has been derived at the integrand level \cite{Mason:2010sr, Caron-Huot:2010ek}, although a rigorous UV regularization for the super-loop has not been carried out \cite{Belitzky:2010de, Beisert:2009mm, Beisert:2010ez, Beisert:2011sf}.

\[
A_n(\lambda_i, \bar{\lambda}_i, \eta_i) = W_n(x_i, \theta_i)(1 + \mathcal{O}(\epsilon)), \quad W_n = \frac{1}{N_c} \langle \text{Tr} \mathcal{P} e^{-\oint A(x_i, \theta_i)} \rangle.
\]

- The chiral super Wilson loop obscures one chiral half of superconformal symmetries. As a natural generalization, Wilson loops in non-chiral $\mathcal{N} = 4$ superspace generally manifest the full symmetry \cite{Caron-Huot:2010ek, Beisert:2011sf, Beisert:2011sf, Beisert:2011sf}. 

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We define \textit{BDS-subtracted} S-matrix: \( A_{n,k} = A_{n,k}^{\text{BDS}} \times R_{n,k} \), which is a finite object depending on dual conformal cross-ratios and the so-called R-invariants. It has simple collinear limits, and by definition, \( R_{4,0} = R_{5,0} = R_{5,1}/R_{5,1}^{\text{tree}} = 1 \).

Such invariants can be constructed using twistors of the dual (super)space \cite{Hodges2009},

- momentum twistor: \( Z_i = (Z_i^a, \chi_i^A) = (\lambda_i^\alpha, x_i^{\alpha\dot{\alpha}} \lambda_i^\alpha, \theta_i^{\alpha A} \lambda_i^\alpha) \);
- four-bracket: \( \langle i j k l \rangle = \epsilon_{abcd} Z_i^a Z_j^b Z_k^c Z_l^d \), e.g. \( u_1 = \frac{\langle 1234 \rangle \langle 4561 \rangle}{\langle 1245 \rangle \langle 3461 \rangle} \);
- R-invariant: \( [i j k l m] = \frac{\delta^{0|4} (\chi_i^A \langle j k l m \rangle + \text{cyclic})}{\langle i j k l \rangle \langle j k l m \rangle \langle k l m i \rangle \langle l m i j \rangle \langle m i j k \rangle} \).

They form the fundamental representation of the dual superconformal algebra,

\[
Q_A^a = (Q_A^\alpha, \bar{Q}_A^{\dot{\alpha}}) = \sum_{i=1}^{n} Z_i^a \frac{\partial}{\partial \chi_i^A}, \quad \bar{Q}_a^A = (\bar{Q}_a^\alpha, \bar{Q}_a^{\dot{\alpha}} = \bar{s}_a^A) = \sum_{i=1}^{n} \chi_i^A \frac{\partial}{\partial Z_i^a},
\]

\[
K_b^a = (P_{\alpha \dot{\alpha}}, \mathcal{R}_{\alpha \dot{\alpha}}, \mathcal{M}_{\alpha \beta}, \bar{M}_{\dot{\alpha} \dot{\beta}}, \mathcal{D}) = \sum_{i=1}^{n} Z_i^a \frac{\partial}{\partial Z_i^b}, \quad R_B^A = \mathcal{R}_B^A = \sum_{i=1}^{n} \chi_i^A \frac{\partial}{\partial \chi_i^B}.
\]
The BDS-subtracted S-matrix is not invariant under the naive $\bar{Q}_a^A$. We propose an all-loop equation for the “anomaly” as collinear integral (see also [Bullimore, Skinner 2011]),

$$\bar{Q}_a^A R_{n,k} = \Gamma_{\text{cusp}} \text{res}_{\epsilon=0} \int_{\tau=0}^{\tau=\infty} \left( d^2 |^3 Z_{n+1} \right)^A_a \left[ R_{n+1,k+1} - R_{n,k} R^{\text{tree}}_{n+1,1} \right] + \text{cyclic},$$

where the cusp anomalous dimension is known $\Gamma_{\text{cusp}} = g^2 - \frac{\pi^2}{3} g^4 + \frac{11 \pi^4}{45} g^6 + \ldots$.

The RHS is an 1d integral over $\tau$; one then computes the residue at $\epsilon \to 0$,

$$Z_{n+1} = Z_n - \epsilon (Z_{n-1} - \frac{\langle n-1 n 23 \rangle}{\langle n 123 \rangle} \tau Z_1) + O(\epsilon^2),$$

$$\text{res}_{\epsilon=0} \int_{\tau=0}^{\tau=\infty} \left( d^2 |^3 Z_{n+1} \right)^A_a = \frac{\langle n-1 n 23 \rangle}{\langle n 123 \rangle} (n-1 n 1)_a \int_{\epsilon=0}^{\infty} \epsilon d\epsilon \int_{0}^{\infty} d\tau (d^0 |^3 X_{n+1})^A_a.$$
The S-matrix from the symmetry: a new proposal

- Using the discrete parity symmetry, we derive an equivalent equation for level-one generator, \( Q^{(1)a}_A = (s^a_A, \ldots) = \frac{1}{2} \sum_{i,j} \text{sgn}(j - i) \left( Z^a_i \frac{\partial}{\partial Z^b_i} Z^b_j \frac{\partial}{\partial \chi^A_j} - Z^a_i \frac{\partial}{\partial \chi^B_i} \chi^B_j \frac{\partial}{\partial \chi^A_j} \right) \),

\[
Q^{(1)a}_A R_{n,k} = \Gamma_{\text{cusp}} Z^n \lim_{\epsilon \to 0} \int_0^\infty \frac{d\tau}{\tau} (d\eta_{n+1})_A \left( R_{n+1,k} - \sum_{i,j} C_{i,j} \frac{\partial R_{n,k}}{\partial \chi^j} \right) + \text{cyclic}.
\]

- The equations essentially amount to Yangian invariance of the S-matrix. RHS are not anomalies: they should be interpreted as quantum corrections of (naive) symmetry generators acting on the S-matrix [Bargheer Beisert Galleas Loebbert McLoughlin 2009] [Sever Vieira 2009] [Beisert Henn McLoughlin Plefka 2010].

- We claim that the equations are valid for any value of the coupling. When expanded in powers of \( \Gamma_{\text{cusp}} \), they recursively give derivatives of all-loop amplitudes.

- The differential equations are nice: both sides are finite, regulator independent, and manifest the transcendentality of loop amplitudes. They are powerful: together with collinear limits, the solutions uniquely determine the full S-matrix.
The S-matrix from the symmetry: outline of a derivation

- The way $\bar{Q}$ acts on a Wilson loop is by inserting a fermion operator on the edges, which was calculated in explicit examples using Feynman diagrams [Caron-Huot 2011]

$$\bar{Q}_\alpha^A \langle W_n \rangle \propto g^2 \oint dx \langle \psi^A + F\theta^A + \ldots \rangle^\alpha W_n \rangle.$$

- The key new ingredient: the fermion insertion is the unique excitation with given quantum numbers. The Operator Product Expansion [Alday Gaiotto Maldacena Sever Vieira 2010] allows us to extract the excited $n$-gon Wilson loop from an $(n+1)$-gon in collinear limit,

$$\frac{1}{A_{n}^{BDS}} \bar{Q} \langle W_{n,k} \rangle = \frac{g^2}{F(g^2)} \text{res}_{\epsilon=0} \int_{\tau=0}^{\tau=\infty} d^2|3| Z_{n+1}^{R_{n+1,k+1}}(\tau, \epsilon) + \text{cyclic}.$$  

Given that BDS ansatz is one-loop exact, we obtain the $\bar{Q}$ of BDS,

$$\langle W_{n,k} \rangle \bar{Q} \frac{1}{A_{n}^{BDS}} = -\Gamma_{\text{cusp}} R_{n,k} \text{res}_{\epsilon=0} \int_{\tau=0}^{\tau=\infty} d^2|3| Z_{n+1}^{R_{n+1,1}^{\text{tree}}}(\tau, \epsilon) + \text{cyclic}.$$  

- Both $\tau$ integrals diverge, but the sum must be finite, so we have $g^2/F(g^2) = \Gamma_{\text{cusp}}$. A crucial test of our derivation is to check the dispersion relation of the insertion.
The S-matrix from the symmetry: outline of a derivation

\[ \psi \quad n \quad n-1 \quad n-2 \quad \ldots \quad 1 \quad 2 \quad \ldots \quad n+1 \quad n \quad n-1 \quad n-2 \quad \ldots \]
The S-matrix from the symmetry: outline of a derivation

- The fermion operators are labeled by a momentum, \( p \), conjugate to its position along the edge. We want to understand the \( \log \epsilon \) term in momentum space,

\[
\lim_{\epsilon \to 0} \log \left( \int_{0}^{\infty} d\tau \tau^{i\frac{p}{2}} d^{0|3} \chi_{n+1} R_{n+1,1} \right) \to \log \epsilon \times \gamma(p) + C(p),
\]

where the dispersion relation \( \gamma(p) \) has to match that of a fermion excitation of the null edge, known for any values of the coupling thanks to integrability [Basso 2010].

- We have derived \( R_{6,1} \) up to two loops, which can be used to give \( \gamma(p) \) to order \( \Gamma_{\text{cusp}}^2 \),

\[
\gamma(p) = \Gamma_{\text{cusp}} (\psi_+ - \psi(1)) - \frac{\Gamma_{\text{cusp}}^2}{8} \left( \psi''_+ + 4\psi'_- (\psi_- - \frac{1}{p}) + 6\zeta(3) \right).
\]

This agrees precisely with [Basso 2010], and it also confirms the prefactor must be \( \Gamma_{\text{cusp}} \).

- For RHS of the equations, we only need the total-\( \tau \) integral (zero-momentum). The cancelation of \( \log \epsilon \) divergences in that case is guaranteed by the Goldstone theorem: the fermion with \( p = 0 \) is a Goldstone fermion, thus \( \gamma(0) = 0 \).
The simplest case, MHV remainder function, $R_{n,0}$, is independent of Grassmann variables. We can obtain all the derivatives from its $\bar{Q}$,

$$\frac{\partial}{\partial \chi_i^a} \bar{Q}^1_i R_{n,0} = \frac{\partial}{\partial Z_i^a} R_{n,0},$$

which uniquely determine $R_{n,0}$, up to a constant (fixed by collinear limit). From the RHS, we can already deduce its total derivative must be of the form

$$dR_{n,0} = \sum_{i,j} F_{i,j} d\log\langle i-1 i i+1 j \rangle,$$

which holds to all loops. This proves the conjecture of [Caron-Huot 2011].

Remarkably, the solution to $\bar{Q}$ equation is also unique for NMHV amplitude, up to a linear combination of R-invariants, which can be fixed by collinear limits.

We need both equations beyond NMHV. For all-loop $N^k$MHV, the solutions are unique, up to invariants under naive $Q$, $\bar{Q}$ and $Q^{(1)}$. It is known [Korchemsky Sokatchev 2010][Drummond Ferro 2010] that all such invariants are given by the Grassmannian formula [Arkani-Hamed Cachazo Cheung Kaplan 2009].
From the collinear integral of \( R_{7,1}^{1\text{-loop}} \), one can easily compute the derivative of two-loop MHV hexagon, reproducing the formula in \([\text{Goncharov Spradlin Vergu Volovich 2010}]\) \([\text{Del Duca Duhr Smirnov 2010}]\)

\[
R_{6,0}^{2\text{-loop}} = 4 \sum_{i=1}^{3} \left( L_4^+(u_i) - \frac{1}{2} \text{Li}_4(1 - \frac{1}{u_i}) \right) - \frac{1}{2} \left( \sum_{i=1}^{3} \text{Li}_2(1 - \frac{1}{u_i}) \right)^2 + \frac{1}{6} J^4 + \frac{\pi^2}{3} J^2 + \frac{\pi^4}{18}.
\]

Higher-point amplitudes are similar; we found the symbol agrees with \([\text{Caron-Huot 2011}]\).

We derived the two-loop NMHV hexagon, and found agreement with results in \([\text{Kosower Roiban Vergu 2011}]\) and \([\text{Dixon Drummond Henn 2011}]\). Similarly we computed the symbol for the heptagon.

An ansatz was proposed for \( S[R_{6,0}^{3\text{-loop}}] \) \([\text{Dixon Drummond Henn 2011}]\), based on physical considerations, e.g. OPE constraints, and assumptions on possible forms of the symbol. We confirmed their assumptions, and fixed the two undetermined parameters,

\[
S[R_{6,0}^{3\text{-loop}}] = \left( S[X] - \frac{3}{8} S[f_1] + \frac{7}{32} S[f_2] \right) (u_1, u_2, u_3).
\]
Amplitudes/Wilson loops simplify significantly for the restricted kinematics when the $2n$ external momenta/edges are embedded in a two-dimensional subspace \cite{AldayMaldacena2009, DelDucaDuhrSmirnov2009, HeslopKhoze2010}. It is natural to do the reduction supersymmetrically, and the symmetry factorizes $PSU(2,2\mid 4) \rightarrow SL(2\mid 2)_{\text{even}} \times SL(2\mid 2)_{\text{odd}}$:

$$Z_{2i-1} = (\lambda_{2i-1}^1, 0, \lambda_{2i-1}^3, 0, \chi_{2i-1}^1, 0, \chi_{2i-1}^3, 0), \quad Z_{2i} = (0, \lambda_{2i}^2, 0, \lambda_{2i}^4, 0, \chi_{2i}^2, 0, \chi_{2i}^4).$$

Four-brackets factorize,

$$\langle 2i-1 2j-1 2k 2l \rangle = \langle 2i-1 2j-1 \rangle [2k 2l];$$

even and odd cross-ratios are built from 1d distances, $u_{a,b,c,d} = \frac{\langle ab \rangle \langle cd \rangle}{\langle ac \rangle \langle bd \rangle}$.

Superamplitudes will be built from “mini” R-invariants in even and odd sector,

$$(a \ b \ c) = \frac{\delta^{0|2} \langle a \ b \rangle \chi_c + \langle b \ c \rangle \chi_a + \langle c \ a \rangle \chi_b}{\langle a \ b \rangle \langle b \ c \rangle \langle c \ a \rangle},$$

Tree amplitudes are trivial combinations of R-invariants, which, e.g. for $N^2$MHV, are products of $(a \ b \ c \ d) := -(a \ b \ c)(a \ c \ d)$. Loop amplitudes are combinations with coefficients being pure, transcendental functions of conformal cross-ratios.
Jumpstarting amplitudes II: restricted kinematics

- The $\bar{Q}$ equation in restricted kinematics is derived by considering the overlap of a $2n$-gon with the collinear limit of $(2n+2)$-gon. In the even sector, we have,

$$
\bar{Q}^A_{\alpha} R_{2n,k} = \Gamma_{\text{cusp}} \int d^{1|2} \lambda_{2n+1} \int d^{0|1} \lambda_{2n+2} (R_{2n+2,k+1} - R_{\text{tree}} R_{2n,k}) + \text{cyclic},
$$

where we take $\lambda_{2n+2} = \lambda_{2n} + \epsilon \lambda_2$ supersymmetrically, and explicitly the measure is

$$
\int d^{1|2} \lambda_{2n+1} \int d^{0|1} \lambda_{2n+2} = \lambda_{2n,a} \lim_{\epsilon \to 0} \int_{\lambda_{2n-1}}^{\lambda_1} \langle \lambda_{2n+1} d\lambda_{2n+1} \rangle \int d^2 \chi_{2n+1} (d\chi_{2n+2})^A.
$$

- From a reasonably nice form of $N^2$MHV tree, we applied the equation twice and derived the $2n$-point two-loop MHV, which agrees with \cite{HeslopKhoze2010, GaiottoMaldacenaSeverVieira2010}.

- A nice byproduct from the computation is the one-loop NMHV, now written in a basis of R-invariants, in terms of functions of cross-ratios, e.g. the octagon

$$
R_{8,1} = ((3 5 7)[2 4 6] f_{8,1}^1 (u_1, u_2) + 7 \text{ cyclic}) + R_{\text{tree}}^1 f_{8,1}^2 (u_1, u_2);
$$

$$
f_{8,1}^{1,\text{loop}} = \log(1-u_1) \log(1-u_2), \quad f_{8,1}^{2,\text{loop}} = \log u_1 (1-u_1) \log u_2 (1-u_2).
$$
Jumpstarting amplitudes II: one-loop $N^2$MHV

- For $k+\ell=3$, i.e. one-loop $N^2$MHV, two-loop NMHV and three-loop MHV, new structures, such as combinations $x-y, 1-x-y$, appear. We computed the amplitudes explicitly using the equations. The result is highly non-trivial and interesting.

- The one-loop $N^2$MHV octagon can be put into a nice form ($u_i := u_{i,i+2,i+4,i+6}$)

$$R_{8,2}^{\text{tree}} = R_{8,2}^{\text{tree}} \frac{u_1 u_2}{1 - u_1 - u_2} \left( f_{8,2}(u_1, u_2) + f_{8,2}(u_2, u_1) \right) + (3 \text{ cyclic}),$$

where $R_{8,2}^{\text{tree}} = (1357)[2468]$, $f_{8,2}(x, y) = \text{Li}_2(x) + \frac{1}{2} \log x \log \left( \frac{1-x}{y} \right) - \frac{\pi^2}{8}$.

- The same pattern also appears in higher-point $N^2$MHV, e.g. the decagon reads,

$$R_{10,2} = (1357)[26810] f_{10,2}^1(u_1, u_6) + (4 \text{ cyclic}) + [(1357)[46810] f_{10,2}^1(u_1, u_4)$$

$$+ (1357)[2468] f_{10,2}^2(u_1, u_2) + 2(1357)[2410][468] f_{8,2}(1-u_1, u_{10}) + (9 \text{ cyclic})] + \ldots,$$

where $\ldots$ denotes remaining $\log \log$ terms with pure $R$ invariants as coefficients;

- $$f_{10,2}^1(x, y) = 2 \frac{xy}{1 - x - y} \left( f_{8,2}(1 - x, 1 - y) - f_{8,2}(y, x) \right),$$

- $$f_{10,2}^2(x, y) = 2 \frac{y(1-x)}{x-y} f_{8,2}(y, 1-x) - 2 \frac{x(1-y)}{x-y} f_{8,2}(x, 1-y).$$
We determined the two-loop NMHV octagon, up to one parameter corresponding to adding a multiple of the one-loop amplitude, in terms of the two functions:

$$f_{8,1}^{1,2\text{-loop}} = \text{Li}_2(1-x, \frac{1-y}{x}) + \text{Li}_2(1-x, \frac{y}{1-x}) - \text{Li}_2(x, \frac{1}{x}) - \text{Li}_2(1, y) + C(x, y) + (x \leftrightarrow y),$$

where the “classical part” $C(x, y)$ involves only polylogarithms of degree 3 or less:

$$C(x, y) = -\left(\text{Li}_3\left(\frac{xy}{(1-x)(1-y)}\right) - \text{Li}_3\left(\frac{x}{1-y}\right) - \text{Li}_3\left(\frac{y}{1-x}\right) + \text{Li}_3(x) + \text{Li}_3(y) + (\text{Li}_2\left(\frac{y}{1-x}\right) - \text{Li}_2(y)) \log \frac{1-y}{x}\right) \log y(1-x)$$

$$+ \left(4\text{Li}_3(y) + 2\text{Li}_3(1-y) + \text{Li}_2(y) \log \frac{x^2(1-y)}{y} - 2\zeta(3)\right) \log(1-x)$$

$$+ \left(\frac{1}{2} \log xy \log(1-x)(1-y) - \frac{1}{2} \log x \log y\right) \log(1-x) \log(1-y)$$

$$+ \frac{1}{2} \text{Li}_2(y) \log^2(1-x) + \frac{3}{2} \text{Li}_2(x) \text{Li}_2(y) + \frac{5}{8} \log^2(1-x) \log^2(1-y),$$

and a simpler function $f_{8,1}^{2,2\text{-loop}} = g(x, y) + (x \leftrightarrow 1-x) + (y \leftrightarrow 1-y) + (x \leftrightarrow y)$:

$$g(x, y) = \left(6\text{Li}_3(1-x) - \text{Li}_2(1-x) \log \frac{1-x}{x} + \log^2 x \log 1-x\right) \log y + \left(\frac{1}{8} \log x + \frac{3}{4} \log 1-x\right) \log x \log^2 y$$

$$- \frac{1}{8} \log x \log 1-x \log y \log 1-y - 3\zeta(3) \log x + \frac{\pi^2}{6} \left(\frac{1}{4} \log x \log \frac{x}{(1-x)} - \log x \log y\right) + \frac{\pi^4}{160}.$$

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The function $f_{8,1}^1$ is basically a component amplitude, $f_{8,1}^1 = \langle 13 \rangle [68] R_{8,1} |\chi^1 \chi^3 \chi^6 \chi^8$. We consider small $x$ expansion, $f_{8,1}^1(v = \frac{x}{1-x}, w = \frac{y}{1-y}) = \sum_{n=1}^{\infty} f_n(w) v^n$:

$$f_{2\text{-loop}}^n = \log v f_{n}^{2\text{-loop}}|_{\log v} + \left[ \frac{w^n}{n^2} (2 \text{Li}_2(-w) + \log w \log(1+w)) \right]_{\text{reg}} + \left[ \frac{2w^n}{n^3} \log \frac{1+w}{w} \right]_{\text{reg}}$$

$$+ \frac{4(-)^n}{n^3} \log(1+w) + \frac{(-)^n}{n} \left( \frac{1}{n} - 2S_1(n) \right) \log w \log(1+w) - \frac{(-)^n}{n^2} \text{Li}_2(-w)$$

$$+ \frac{4(-)^n}{n} (S_1(n) - \frac{1}{n}) \log(1+w)^2 - \frac{(-)^n}{n} (6 \text{Li}_3(-w) - \log w \text{Li}_2(-w) + \pi^2 \log(1+w)),$$

$$f_{n}^{2\text{-loop}}|_{\log v} = \left[ \frac{w^n}{n^2} \log \frac{w}{1+w} \right]_{\text{reg}} + \frac{(-1)^n}{n} \log^2(1+w) - \frac{(-1)^n}{n^2} \log(1+w),$$

where the $\log v$ part agrees with OPE leading-order predictions. The most interesting part is in terms which mix $v$ with $w$, while the remaining terms are factorized.

The result becomes remarkably simple after doing a Fourier (Mellin) transform,

$$f(p, q) = \int_0^1 \frac{dv}{v} \int_0^1 \frac{dw}{w} f(v, w) v^{\frac{p}{2}} w^{\frac{q}{2}};$$

$$f_{8,1}^{1,2\text{-loop}}(p, q) = \frac{\pi}{p \sinh(\frac{\pi p}{2})} \frac{\pi}{q \sinh(\frac{\pi q}{2})} \frac{\coth(\frac{\pi p}{2}) - \coth(\frac{\pi q}{2})}{p - q} + \text{factorized.}$$
• We also derived the two-loop NMHV decagon, whose non-trivial, mixed part is essentially a sum of octagons. Based on this, we obtained the complete function for the three-loop MHV octagon, up to two constants multiplying two-loop MHV and NMHV octagons. All other beyond-the-symbol ambiguities were fixed.

• The result, in terms of functions like $\text{Li}_{3,3}$, is relatively involved, but the small $x$ expansion is compact; in particular the mixed part is similar to two-loop NMHV,

$$
\begin{align*}
  f_{n}^{3\text{-loop}} &= \sum_{i=1}^{n} \left[ c_i w^i \left( \log v \log \frac{w}{1+w} + 2 \, \text{Li}_2(-w) + \log w \log(1+w) \right) + c_i' w^i \log \frac{w}{1+w} \right]_{\text{reg}} \\
  &+ \sum_{i=1}^{n} \left[ \frac{c_i}{w^i} \left( \log v \log(1+w) + 2 \, \text{Li}_2(-w) + \log w \log(1+w) \right) + \frac{c_i'}{w^i} \log(1+w) \right]_{\text{reg}} \\
  &+ \text{factorized}.
\end{align*}
$$

• We expect it to have a nice Mellin representation, and possibly also for higher points. We have a rich set of data: non-trivial but simple, suggesting some underlying picture. How to understand such nice structures from integrability?
Summary and outlook

- The all-loop S-matrix in planar $\mathcal{N} = 4$ SYM is invariant under a suitably deformed Yangian symmetry at the quantum level, and is fully determined by it.

- We derived new, elegant equations based on the quantum-corrected symmetry, and tested them extensively against e.g. results of multi-loop amplitudes and OPE.

- The equations have provided new data for the S-matrix of planar $\mathcal{N} = 4$ SYM; we hope that they will provide more insights into its integrability.

Open questions

- OPE interpretations of the result, especially how to understand multi-particle states? Relations to the spin chain picture in [Sever Wang Vieira 2012]?

- Understanding the equations at strong coupling? Relations to TBA, Y-system?

- Beyond amplitudes in $\mathcal{N} = 4$ SYM: non-chiral Wilson loops/correlation functions in the light-cone limit? the S-matrix of super Chern-Simons from symmetries?