On the integrability of 2D models with $U(N)$ symmetry

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Spectral problem and AdS/CFT correspondence

- Supersymmetric Conformal Field Theory
  - $\mathcal{N} = 4$ 4D SYM theory
  - $\mathcal{N} = 6$ 3D ABJM theory
- Two dimensionless parameters: the 't Hooft coupling κ and the number of colors N_c
- Observables: spectrum of scaling dimensions Δ

AdS/CFT correspondence:

(Planar) CFT is equivalent to (free) superstring on AdS background

$$\text{string tension} \sim \kappa \qquad \text{string coupling} \sim 1/N_c$$

Dictionary:

Spectrum of (planar) scaling dimensions = spectrum of energies of (free) string

Spectral problem and integrability

Main difficulty: How to confront the gauge and string theory?

- Gauge theory is tractable at weak coupling: $\kappa \ll 1$
- String theory is tractable at strong coupling: $\kappa \gg 1$

Need control on the weak/strong coupling interpolation → need non-perturbative methods

Important development: Discovery of integrable structures (in the planar limit)
[Minahan,Zarembo'02],[Beisert,Staudacher'03'05],[Gromov,Vieira'09]
[Lipatov'98],[Braun,Derkachov,Korchemsky,Manashov'98],[Belitsky'99]
[Bena,Polchinski,Roiban'03],[Kazakov,Marshakov,Minahan,Zarembo'04]
[Gromov,Kazakov,Vieira'09],[Gromov,Kazakov,Kozak,Vieira'09],
[Bombardelli,Fioravanti,Tateo'09],[Arutyunov,Frolov'09]

→ Complete solution to spectral problem in the planar limit (conjecture)

Questions

- Are the conjectured equations correct?
- Is the (super) string sigma model on AdS quantum integrable?

Remarks (answers?):

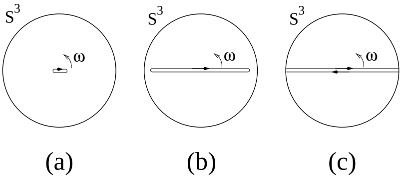
- Tremendous evidence for AdS_5/CFT_4 theory (complete proof?)
- Much less evidence for AdS_4/CFT_3 theory (to my knowledge)

Purpose of this talk:

- Hint at the non-perturbative integrability of string sigma model in $AdS_4 \times CP^3$
- Set the stage for a non-perturbative test of the all-loop ABA equations for ABJM theory

The GKP string

Folded string rotating in AdS_3 with spin S [Gubser,Klebanov,Polyakov'02]



- (a) Short string : $S \sim 0 \longrightarrow$ length $\sim S^{1/2} \sim 0$
- (c) Long string : $S \sim \infty \longrightarrow$ length $= 2\log S \gg 1\, +$ worldsheet homogeneous

Interesting background:

- Vacuum energy density = cusp anomalous dimension
- Non-pertubative test of integrability and of its asymptotic Bethe ansatz (ABA) equations
- Dynamics on GKP is relevant to scattering amplitudes

[Alday,Gaiotto,Maldacena,Sever,Vieira'10'11]
[Caron-Huot,He'11]

Spectrum of excitations in $AdS_5 \times S^5$

Quadratic fluctuations: (relativistic spectrum) [Frolov,Tseytlin'02]

- 5 massless bosons from fluctuations in S^5
- 2 bosons with mass $\sqrt{2}$ for $\perp$ fluctuations in AdS_5/AdS_3
- 1 boson with mass 2 for $\perp$ fluctuation in AdS_3
- 8 fermions with mass 1

Low-energy effective theory: $O(6)$ non-linear sigma model [Alday,Maldacena'07]

- Non-critical sigma model: $\beta \neq 0$
- Dynamical transmutation: mass scale $\sim \Lambda \sim \exp(-\pi\kappa)$
- The theory is integrable and its S-matrix is known exactly

Spectrum of excitations in $AdS_4 \times CP^3$

Quadratic fluctuations: [Alday,Arutyunov,Bykov'09]

- Massless modes : 6 Goldstone bosons from CP^3 and 1 massless Dirac fermion
- 6 real fermions with mass 1
- 2 massive bosons with mass $\sqrt{2}$ and 2

Low-energy effective theory: CP^3 sigma model + fermion [Bykov'10]

$$\mathcal{L} = \kappa(\partial_\mu - iA_\mu)\bar{z}(\partial^\mu + iA^\mu)z + i\bar{\psi}\gamma^\mu(\partial_\mu - 2iA_\mu)\psi + \frac{1}{4\kappa}\left(\bar{\psi}\gamma_\mu\psi\right)^2$$

where $z = (z_1, z_2, z_3, z_4)$ and $\bar{z}z = 1$

Questions:

- What is the physics of the model?
- Is the theory integrable?

Outline

- Fermionic model with $U(N)$ symmetry
- Bosonic dual model
- Integrability
- String model and finite-volume effects

Fermionic model with $U(N)$ symmetry

$U(N)$ fermionic model

CP^{N-1}+fermion:

$$\mathcal{L} = \kappa(\partial_\mu - iA_\mu)\bar{z}(\partial^\mu + iA^\mu)z + i\bar{\psi}\gamma^\mu(\partial_\mu - ikA_\mu)\psi - \frac{\lambda}{2}\left(\bar{\psi}\gamma_\mu\psi\right)^2$$

Ingredients:

- N complex scalar fields z subject to $\bar{z}z = 1$
- Abelian gauge field A_μ with no kinetic term
- 1 massless fermion ψ with charge k

Parameters (couplings):

- Effective string tension κ
- Thirring coupling λ
- Fermion charge k

$U(N)$ fermionic model

CP^{N-1} + fermion:

$$\mathcal{L} = \kappa(\partial_\mu - iA_\mu)\bar{z}(\partial^\mu + iA^\mu)z + i\bar{\psi}\gamma^\mu(\partial_\mu - ikA_\mu)\psi - \frac{\lambda}{2}\left(\bar{\psi}\gamma_\mu\psi\right)^2$$

Symmetries:

- $U(1)$ gauge symmetry
- Global $U(1) \times SU(N)$
- Axial $U(1)$ (classical only)
- Conformal symmetry (classical only)

Remark: No SUSY for generic $N, \ldots$

Questions:

- Physics? clarified using large N expansion [D'Adda,Lüscher,DiVecchia'78],[Witten'78]
- Integrability? for any k and λ?

See also Dmitry's talk!

Related model

Model with N fermions: [Köberle,Kurak'82]

$$\mathcal{L} = \kappa(\partial_\mu - iA_\mu)\bar{z}(\partial^\mu + iA^\mu)z + i\bar{\psi}\gamma^\mu(\partial_\mu - iA_\mu)\psi$$

proposed to be integrable and S-matrix has been identified using large N techniques

Relation: Previous model as minimal and refined version

- Minimal: only 1 (massless) fermion is active when minimally coupled to gauge field [Callan,Dashen,Gross'77]
- Refined: models without Thirring coupling are not renormalizable
- Previous model is more generic: two extra parameters k and λ

Renormalization

- Look at Gauss law

$$J_\mu = i\kappa \bar{z} D_\mu z = k \bar{\psi}\gamma_\mu \psi$$

- Identity between two conserved currents $\rightarrow$ charge k has to be finite

Thirring coupling:

- Look at correlator

$$< \bar{\psi}\gamma_\mu\psi(x)\bar{\psi}\gamma_\nu\psi(0) >$$

- One-loop computation $\rightarrow$ finite only if λ runs

$$\mu\frac{\partial\lambda}{\partial\mu} = \frac{k^2}{4\pi\kappa^2} + \ldots , \qquad \mu\frac{\partial\kappa}{\partial\mu} = \frac{N}{2\pi} + \ldots$$

RG-invariant parameters:

- Dynamical scale Λ

$$\kappa = \frac{N}{2\pi}\log(\mu/\Lambda) + \ldots$$

- UV value of Thirring coupling λ_∞

$$\lambda = \lambda_\infty - \frac{k^2}{2N\kappa} + \ldots$$

- Charge k

Bosonic dual model

Bosonic model

Sigma model: [Campostrini,Rossi'93],[Azaria,Lecheminant,Mouhanna'95]

$$\mathcal{L} = \frac{R^2}{4\pi}(\partial_\mu - iB_\mu)\bar{z}(\partial^\mu + iB^\mu)z + \frac{r^2}{4\pi}B_\mu B^\mu$$

where $\bar{z}z = 1$ and $B_\mu = i\bar{z}\partial_\mu z$

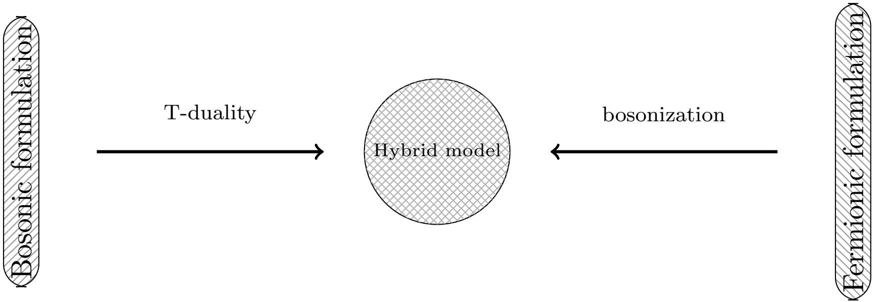
Advantages:

- Computationally easier
- Nice geometrical picture

Geometrically: circle fibered over CP^{N-1}

- $U(1) \times SU(N)$ symmetry
- No gauge symmetry for $r \neq 0$
- At $r = 0$ recover CP^{N-1} model
- Symmetry enhancement at $r = R$: equivalent to $O(2N)$ sigma model

T-duality and bosonization



T-duality

Local coordinates: $z = e^{i\vartheta} z'$

$$B_\mu = -\partial_\mu \vartheta + b_\mu$$

with $\vartheta \sim \vartheta + 2\pi$ parameterizing the circle

$\mathbb{Z}_k$ quotient: $\vartheta \to \vartheta / k$

T-dualization:

$$\mathcal{L}_{fiber} = \frac{r^2}{4\pi k^2}(\partial_\mu \vartheta - k b_\mu)^2 \to \frac{k^2}{r^2}\partial_\mu \varphi \partial^\mu \varphi + \frac{k\varphi}{\sqrt{\pi}}\epsilon^{\mu\nu}\partial_\mu B_\nu$$

where $\varphi \sim \varphi + \sqrt{\pi}$

Complete T-dual Lagrangian:

$$\mathcal{L} = \frac{R^2}{4\pi} D_\mu \bar{z} D^\mu z + \frac{R^2 - r^2}{r^2 R^2} k^2 \partial_\mu \varphi \partial^\mu \varphi + \frac{k\varphi}{\sqrt{\pi}}\epsilon^{\mu\nu}\partial_\mu A_\nu$$

with dummy gauge field A_μ

Bosonization

Main identity: [Coleman'74]

$$\bar{\psi}\gamma_\mu\psi = \frac{1}{\sqrt{\pi}}\epsilon_{\mu\nu}\partial^\nu\varphi$$

with φ the bosonized fermion

Bosonized Lagrangian:

$$\mathcal{L} = \kappa D_\mu \bar{z} D^\mu z + \frac{1+\lambda/\pi}{2}\partial_\mu\varphi\partial^\mu\varphi + \frac{k\varphi}{\sqrt{\pi}}\epsilon^{\mu\nu}\partial_\mu A_\nu$$

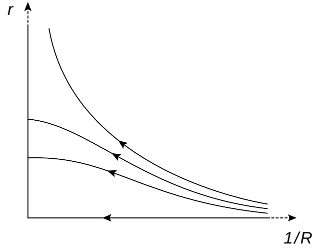
Relations between couplings:

$$R^2 = 4\pi\kappa\,, \qquad \frac{1+\lambda/\pi}{2} = \frac{k^2}{r^2} - \frac{k^2}{R^2}$$

The fermionic and bosonic theories are dual (equivalent)

UV physics

Asymptotically free domain $0 \leqslant r \leqslant R$ [Azaria,Lecheminant,Mouhanna'95]



Running couplings:

$$R^2 = 2N \log(\mu/\Lambda) + 2(2-p) \log\log(\mu/\Lambda) + o(1)$$

$$\frac{1}{r^2} = \frac{1}{2Np} + \frac{N-1}{NR^2} - \frac{(N-1)(2-p)}{NR^4} + O(1/R^6)$$

Two RG invariants:

- Dynamical scale Λ
- UV radius of the circle $p = r^2(\mu = \infty)/2N$

IR physics from large N

[Campostrini,Rossi'93],[Azaria,Lecheminant,Mouhanna'95]

- Very close to $O(2N)$: $2N$ particles (spinons) with mass $m = \Lambda$
- Weakly coupled at large N
- Repulsive scalar interaction as in $O(2N)$ model
- Extra ingredient: attractive interaction mediated by gauge field

$$\langle A^\mu A^\nu \rangle (k^2) = \frac{-i\pi}{N\left(A(k^2) - p\right)} \left(\eta^{\mu\nu} - \frac{k^\mu k^\nu}{k^2}\right) + \frac{i\pi k^\mu k^\nu}{Npk^2}\,,$$

- $p = 0$: linear Coulomb potential and confinement of spinons
  [D'Adda,Lüscher,DiVecchia'78],[Witten'78]
- $p \neq 0$: gauge field has a mass and the potential is screened, spinons are free
- $p = 1$ separates the attractive and repulsive regime

Remark: Same large N physics as fermionic model [D'Adda,Lüscher,DiVecchia'78],[Witten'78]

Integrability

Particular case $N = 2$

Anisotropic $SU(2)$ Principal Chiral Field: [Wiegmann'85],[Polyakov,Wiegmann'83]

$$\mathcal{L}_{N=2} = -\frac{\kappa}{2}\text{tr}\, j_\mu j^\mu + \frac{\kappa\eta}{4}\left(\text{tr}\, j_\mu \sigma_3\right)^2$$

where

$$\Omega = \begin{pmatrix} z_1 & -\bar{z}_2 \\ z_2 & \bar{z}_1 \end{pmatrix}, \qquad j_\mu = \Omega^{-1}\partial_\mu \Omega$$

Symmetries:

- $SU(2) \times U(1)$ for generic η
- $SU(2) \times SU(2) \simeq O(4)$ for $\eta = 0$
- Equivalent to $O(3)$ sigma model for $\eta = 1$

Integrability: See Kentaroh's talk! and Io's and Takuya's posters!

- Integrable for any η [Wiegmann'85],[Balog,Forgacs,Laszlo'00], [Kawaguchi,Matsumoto,Yoshida'12]
- Exact S-matrix

$$\mathbb{S} = \mathbb{S}_{SU(2)} \otimes \mathbb{S}^{(p)}_{Sine-Gordon}$$

Classical integrability

Lax connection:

- Look for flat and conserved current $\tilde{\jmath}_\mu$

$$\partial_\mu \tilde{\jmath}^\mu = 0\,, \qquad \partial_\mu \tilde{\jmath}_\nu - \partial_\nu \tilde{\jmath}_\mu + \left[\tilde{\jmath}_\mu, \tilde{\jmath}_\nu\right] = 0$$

- Lax connection

$$L_\mu = \frac{1}{1-x^2}\tilde{\jmath}_\mu + \frac{x}{1-x^2}\epsilon_{\mu\nu}\tilde{\jmath}^\nu$$

  will be flat for any value of the spectral parameter x

- Generate infinitely many conserved charges

$$M(x) = P\exp\int d\sigma\, L_\sigma(x)$$

Results:

- Impossible for bosonic model at generic value of deformation parameter η if $N > 2$
- Always possible for fermionic model for any value of deformation parameter p

Quantum/classical mixing

Puzzle: F model classically integrable while B model is not

Resolution:

- Classical integrability of F model relies on the axial conservation law

$$\epsilon^{\mu\nu}\partial_\mu\bar{\psi}\gamma_\nu\psi = 0$$

- Axial anomaly is incoded in B model at the classical level and integrability is lost
- Bosonization is not a classical transformation

What does it imply?

- Relate couplings:

$$\eta = 1 - \frac{pN}{2\pi\kappa} + \ldots$$

- Cannot distinguish between $p = 0$ and $p \neq 0$ in B model classically
- Classically B model probes regime $p \sim \kappa \gg 1$

Quantum integrability

- Quantize previous flat current: cumbersome
- Alternative: counting argument [Goldschmidt,Witten'78]

Argument:

- Conformal invariance at classical level
- Higher-spin conservent currents

$$\partial_+ T^n_{--} = 0$$

- Conformal invariance broken at quantum level by anomaly (running of couplings)

$$\partial_+ T^2_{--} = A$$

  with A dim $= 5$ and spin $= 3$
- If A can be written as

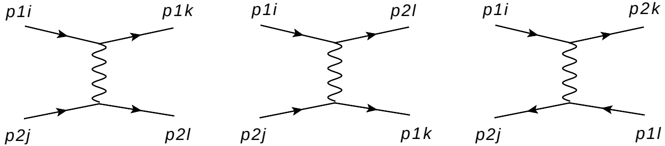
$$A = \partial_+ F_1 + \partial_- F_2$$

  then higher conservation law is deformed but not spoiled

Counting and fine-tuning:

- 1 unmatched anomaly (for $N > 2$ otherwise none)
- 1 free parameter $p \longrightarrow$ expect integrability is restored by fine tuning p

Tree level large N amplitudes



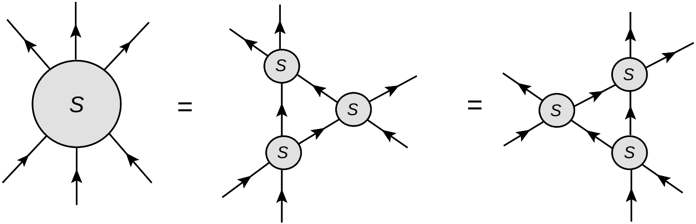
$$u_1(\theta) = 1 - \frac{i\pi}{N}\left(\frac{2m^2}{\sqrt{-st}} + \frac{s-2m^2}{p\sqrt{-st}}\right)$$

$$u_2(\theta) = \quad -\frac{i\pi}{N}\left(\frac{1}{\theta} + \frac{s}{\theta s + 2(p-1)\sqrt{-st}}\right)$$

$$r_1(\theta) = \quad -\frac{i\pi}{N}\left(\frac{1}{\theta} - \frac{s}{\theta s + 2(p-1)\sqrt{-st}}\right)$$

where $s = 4m^2 \cosh^2(\theta/2)$ and $t = 4m^2 - s$

Factorized scattering



Factorization of the scattering: [Berg,Karowski,Weisz,Kurak'77]

$$u_2(\theta) = -\frac{i\nu}{\theta} u_1(\theta)$$

for some constant ν

Consequence: only two solutions

- $p = \infty$: minimal $O(2N)$ S-matrix
- $p = 1$: minimal $U(N)$ reflectionless S-matrix

Agree with findings of [Köberle,Kurak'82]

Finite N analysis

Test: Free energy density $f(h)$ at given chemical potential h

- QFT approach: $h \gg \Lambda \to f(h, \Lambda, p)$
- S-matrix (exact) approach: $h \gg m$ at given p (where integrability is found)

QFT:

- Classically

$$z_1 = e^{-i\omega\tau}\,, \qquad z_j = 0\,, \qquad j = 2, \dots, N$$

- One-loop free energy

$$f(h) = -\frac{ph^2N}{2\pi}\left[1 - \frac{p(N-1)}{N\log(h/\Lambda)} - \frac{p(N-1)(N+p-2)\log\log(h/\Lambda)}{N^2\log^2(h/\Lambda)} + \dots\right]$$

S-matrix:

$$f(h) = -\frac{h^2N}{2\pi}\left[1 - \frac{N-1}{N\log(h/m)} - \frac{(N-1)^2\log\log(h/m)}{N^2\log^2(h/m)} + \dots\right]$$

Comparison:

- Test of the candidate S-matrix at finite N: OK $\to p = 1$ confirmed
- Test of the renormalizability of the Thirring coupling: OK

String model and finite-volume effects

String model

Lagrangian: [Bykov'10]

$$\mathcal{L} = \kappa(\partial_\mu - iA_\mu)\bar{z}(\partial^\mu + iA^\mu)z + i\bar{\psi}\gamma^\mu(\partial_\mu - 2iA_\mu)\psi + \frac{1}{4\kappa}\left(\bar{\psi}\gamma_\mu\psi\right)^2$$

Is it integrable?

- Recall that
$$\lambda = \pi\left(\frac{k^2}{Np} - 1\right) - \frac{k^2}{2N\kappa} + \ldots = -\frac{1}{2\kappa} + \ldots$$
for $N = 4, k = 2$ and $p = 1$
- Parameters are fine tuned as it should for integrability!

Questions: Energy spectrum in finite volume (i.e., for worldsheet = cylinder)?

Energy levels in finite volume

ABA equations:

- S-matrix for spinons in symmetric channel: $S(\theta)$
- Boundary conditions for multi-spinon wave-function: twist q

$$e^{-ip(\theta_k)L} = q \prod_{j\neq k}^{K} S(\theta_k - \theta_j)$$

Discrete symmetries: [Witten'78]

- Axial $U(1)$ symmetry broken down to $\mathbb{Z}_{2k}$ by the anomaly
- Remnant $\mathbb{Z}_{2k}$ symmetry is spontaneously broken down to $\mathbb{Z}_2$
- $\mathbb{Z}_k \simeq \mathbb{Z}_{2k}/\mathbb{Z}_2 \rightarrow k$ vacua
- Spinons are solitons interpolating between adjacent vacua

Consequences: The charge k plays an important role in finite volume (restoration $\mathbb{Z}_k$ symmetry)

- Hilbert space splits into k different sectors (Bloch waves)
- Selection rule (Gauss law): $K - \bar{K} = kF$ with F the fermion number
- Need a twist $q = ?$

Useful analogy

The k-folded sine-Gordon: [Zamolodchikov'94],[Bajnok,Palla,Takacs,Wagner'00]

$$\mathcal{L}_{SG} = \frac{2\pi k^2}{\beta^2}\partial_\mu\varphi\partial^\mu\varphi + \frac{m_0^2}{\beta^2}\cos\left(2k\sqrt{\pi}\varphi\right)$$

with $\varphi \sim \varphi + \sqrt{\pi}$

- k vacua (not all degenerate in finite volume)
- $\mathbb{Z}_k$ symmetry $\rightarrow$ k sectors (Bloch waves)
- twist in ABA equations (for boundary conditions $\varphi(\sigma + L) = \varphi(\sigma) \mod \sqrt{\pi}$)

$$q^k = 1$$

- selection rule: $K - \bar{K} = kn$ with n integer (= winding number)

Refined analysis

Question: Should we conclude that

$$q^{k} = 1$$

for the (dual) fermionic model as well?

Answer: No;

Theories can be equivalent in infinite volume but have different finite-volume spectra

Example: Sine-Gordon versus Massive Thirring model [Klassen,Melzer'92]

Hint:

- Theories have different sets of local operators
- This difference maps to the spectra by state/operator correspondence
- Fermionic theory contains operators that anticommute with the $\mathbb{Z}_2$ operation $(-1)^F$

$$\text{e.g.,} \qquad \psi_{\pm} z_i z_j \, |\text{vacuum}\rangle$$

Proposal: twist for the spectum in the Neveu-Schwarz sector

$$q^{k} = (-1)^F$$

ABA equations for string model

$$e^{-ip(\theta_k)L} = q \prod_{j\neq k}^{K} S(\theta_k - \theta_j) \prod_{j=1}^{\bar{K}} t_1(\theta_k - \bar{\theta}_j) \prod_{j=1}^{K_1} \frac{2\theta_k/\pi - u_{1,j} + \frac{i}{2}}{2\theta_k/\pi - u_{1,j} - \frac{i}{2}},$$

$$\prod_{j=1}^{K} \frac{u_{1,k} - 2\theta_j/\pi + \frac{i}{2}}{u_{1,k} - 2\theta_j/\pi - \frac{i}{2}} = \prod_{j\neq k}^{K_1} \frac{u_{1,k} - u_{1,j} + i}{u_{1,k} - u_{1,j} - i} \prod_{j=1}^{K_2} \frac{u_{1,k} - u_{2,j} - \frac{i}{2}}{u_{1,k} - u_{2,j} + \frac{i}{2}},$$

$$1 = \prod_{j\neq k}^{K_2} \frac{u_{2,k} - u_{2,j} + i}{u_{2,k} - u_{2,j} - i} \prod_{j=1}^{K_3} \frac{u_{2,k} - u_{3,j} - \frac{i}{2}}{u_{2,k} - u_{3,j} + \frac{i}{2}} \prod_{j=1}^{K_1} \frac{u_{2,k} - u_{1,j} - \frac{i}{2}}{u_{2,k} - u_{1,j} + \frac{i}{2}},$$

$$\prod_{j=1}^{\bar{K}} \frac{u_{3,k} - 2\bar{\theta}_j/\pi + \frac{i}{2}}{u_{3,k} - 2\bar{\theta}_j/\pi - \frac{i}{2}} = \prod_{j\neq k}^{K_3} \frac{u_{3,k} - u_{3,j} + i}{u_{3,k} - u_{3,j} - i} \prod_{j=1}^{K_2} \frac{u_{3,k} - u_{2,j} - \frac{i}{2}}{u_{3,k} - u_{2,j} + \frac{i}{2}},$$

$$e^{-ip(\bar{\theta}_k)L} = 1/q \prod_{j\neq k}^{\bar{K}} S(\bar{\theta}_k - \bar{\theta}_j) \prod_{j=1}^{K} t_1(\bar{\theta}_k - \theta_j) \prod_{j=1}^{K_3} \frac{2\bar{\theta}_k/\pi - u_{3,j} + \frac{i}{2}}{2\bar{\theta}_k/\pi - u_{3,j} - \frac{i}{2}}$$

with $K - \bar{K} = 2F$ and $q^2 = (-1)^F$

Conclusion

- Evidence for quantum integrability of a family of $U(N)$ models
- For $N = 4$ it applies to string sigma model in $AdS_4 \times CP^3$ in AM decoupling limit
- Non-perturbative test of the quantum integrability of the complete string sigma model

Open questions:

- Can we write down complete TBA equations for the model and compute energy of vacua?
- Can we match the gauge-theory ABA equations with the ones for the Bykov's model?

Thank you!