Exact Results on the ABJM Fermi Gas

Kazumi Okuyama (Shunshu U) w/ Y. Hatsuda (TIT → DESY), S. Moriyama (Nagoya U) [arXiv:1207.4283]

ABJM Matrix Model

- Partition function of $U(N)_k \times U(N)_{-k}$ ABJM theory on S^3

$$Z_N = \frac{1}{(N!)^2}\int \frac{d^N x d^N y}{(2\pi)^{2N}} \left[\frac{\prod_{i<j} 2\sinh\frac{x_i-x_j}{2}\cdot 2\sinh\frac{y_i-y_j}{2}}{\prod_{i,j} 2\cosh\frac{x_i-y_j}{2}}\right]^2 e^{\frac{ik}{4\pi}(x^2-y^2)}$$

- Mirror description

$$Z_N = \frac{1}{2^N N!}\int \prod_i \frac{dx_i}{2\pi k}\frac{1}{2\cosh\frac{x_i}{2}}\cdot\prod_{i<j}\tanh^2\frac{x_i-x_j}{2k}$$

Grand Canonical Partition Function

- Grand partition function is written as a Fredholm determinant

$$\Xi(\mu) = \sum_{N=0}^{\infty} e^{N\mu} Z_N = \mathrm{Det}(1+e^{\mu-H})$$

- One-body Hamiltonian of Fermi gas [Marino-Putrov]

$$e^{-H} = \sqrt{c_x}\, c_p \sqrt{c_x}$$

where

$$c_x = (2\cosh x/2)^{-1}, \quad [x,p] = i\hbar, \quad \hbar = 2\pi k$$

- We will concentrate on the $k=1$ case
  - ▷ $k=1$ is special (SUSY enhancement from $\mathcal{N}=6$ to $\mathcal{N}=8$)
  - ▷ Truly eleven-dimensional dual $AdS_4\times S^7$
  - ▷ Hidden integrability?

Energy Spectrum of Fermi Gas

- Take the monomial basis

$$\psi_n(x) = (t_x)^n c_x^{3/2}, \quad t_x = \tanh\frac{x}{2}$$

- Hamiltonian in the monomial basis

$$e^{-H}\psi_n = \sum_{m=0}^{\infty} M_{n,m}\psi_m$$

- M is a Hankel matrix given by the Catalan number $C_n = \frac{(2n)!}{n!(n+1)!}$

$$M_{n,m} = \begin{cases} \frac{1}{2^{n+m+3}} C_{\frac{n+m}{2}} & (n+m=\text{even}) \\ 0 & (n+m=\text{odd}) \end{cases}$$

- Numerical result of the eigenvalues of the Hamiltonian H

$$E_n^2 \approx \frac{\pi^2}{2}n + E_0^2 \quad (n=0,1,\cdots), \qquad E_0 = 1.975758\ldots$$

Density Matrix $\rho(x,x') = \langle x|e^{-H}|x'\rangle$

- Partition function is completely determined by the even part ρ_+

$$\rho_\pm(x,x') = \frac{\rho(x,x')\pm\rho(x,-x')}{2}$$

- ρ_+^n is given by a series of functions $\phi_s(x)$ [Tracy-Widom]

$$\rho_+^{2n+1}(x,x') = \frac{(c_x c_{x'})^{-1/2}}{4(\cosh x + \cosh x')}\sum_{s=0}^{2n}(-1)^s\phi_s(x)\phi_{2n-s}(x')$$

$$\rho_+^{2n}(x,x') = \frac{(c_x c_{x'})^{-1/2}}{4(\cosh x - \cosh x')}\sum_{s=0}^{2n-1}(-1)^s\phi_s(x)\phi_{2n-1-s}(x')$$

- Recursion relation for $\phi_s(x)$ and $\widetilde{\phi}_s(p) = \int\frac{dx}{2\pi}e^{-i\frac{px}{2\pi}}\phi_s(x)$

$$\widetilde{\phi}_{2n+1}(p) = \frac{1}{2}c_p\sum_{s=0}^{2n}(-1)^s\phi_s(p)\phi_{2n-s}(0)$$

$$\widetilde{\phi}_{2n}(p) = \frac{1}{2}s_p^2 c_p^{-1}\sum_{s=0}^{2n-1}(-1)^s\phi_s(p)\phi_{2n-1-s}(0)$$

Anomalous Commutation Relation

- Quantum mechanics at $k=1$

$$[x,p] = 2\pi i$$

- Commutation relation for $c_x = (2\cosh\frac{x}{2})^{-1}$ and $s_x = (2\sinh\frac{x}{2})^{-1}$

$$c_p c_x = -i s_x \Pi s_p, \quad c_x c_p = i s_p \Pi^\dagger s_x$$

- Π is a projector

$$\Pi = 1 - |0_p\rangle\langle 0_x|, \quad \Pi^2 = \Pi$$

- From this relation, we find a non-trivial identity

$$\sum_{s=0}^{\infty} z^s\phi_s(0) = \frac{\mathrm{Det}(1+z\rho_-)}{\mathrm{Det}(1-z\rho_+)}$$

Exact Partition Function Z_N at $k=1$

- We obtained Z_N up to $N=9$:

$$Z_0 = 1, \qquad Z_1 = \frac{1}{2^2}$$

$$Z_2 = \frac{1}{2^4\pi}, \qquad Z_3 = \frac{\pi-3}{2^6\pi}$$

$$Z_4 = \frac{-\pi^2+10}{2^{10}\pi^2}, \qquad Z_5 = \frac{-9\pi^2+20\pi+26}{2^{12}\pi^2}$$

$$Z_6 = \frac{36\pi^3-121\pi^2+78}{2^{14}3^2\pi^3}, \qquad Z_7 = \frac{-75\pi^3+193\pi^2+174\pi-126}{2^{16}3\pi^3}$$

$$Z_8 = \frac{1053\pi^4-2016\pi^3-4148\pi^2+876}{2^{21}3^2\pi^4}$$

$$Z_9 = \frac{5517\pi^4-13480\pi^3-15348\pi^2+8880\pi+4140}{2^{23}3^2\pi^4}$$

▷ See also [Putrov-Yamazaki]
They found Z_N up to $N=19$ from TBA

$$Z_{10} = \frac{-81000\pi^5+207413\pi^4+190800\pi^3-136700\pi^2+16860}{2^{25}3^2 5^2\pi^5}$$

$$Z_{11} = \frac{447525\pi^5-1091439\pi^4-1289300\pi^3+837300\pi^2+381900\pi-122580}{2^{27}3^2 5^2\pi^5}$$

Non-Perturbative Corrections

- Perturbative part = Airy function [Fuji-Hirano-Moriyama]

$$Z_{\mathrm{Airy}} = \left(\frac{\pi^2 k}{2}\right)^{\frac{1}{3}} e^{A(k)}\cdot\mathrm{Ai}\left[\left(\frac{\pi^2 k}{2}\right)^{\frac{1}{3}}\left(N-\frac{k}{24}-\frac{1}{3k}\right)\right]$$

- Non-perturbative part of free energy

$$F_{\mathrm{np}} = F_{\mathrm{exact}} - F_{\mathrm{Airy}} = -\log Z_{\mathrm{exact}} + \log Z_{\mathrm{Airy}}$$

- For general k, there are two types of instanton corrections

$$\text{membrane instanton} \sim e^{-\pi\sqrt{2kN}}$$

$$\text{worldsheet instanton} \sim e^{-2\pi\sqrt{\frac{2N}{k}}}$$

- We find the membrane instanton $e^{-\pi\sqrt{2kN}}$ is absent when $k=1$

$$F_{\mathrm{exact}} - F_{\mathrm{Airy}} \sim e^{-2\pi\sqrt{2N}}$$