Fundamental mathematical structures in statistical and quantum systems.

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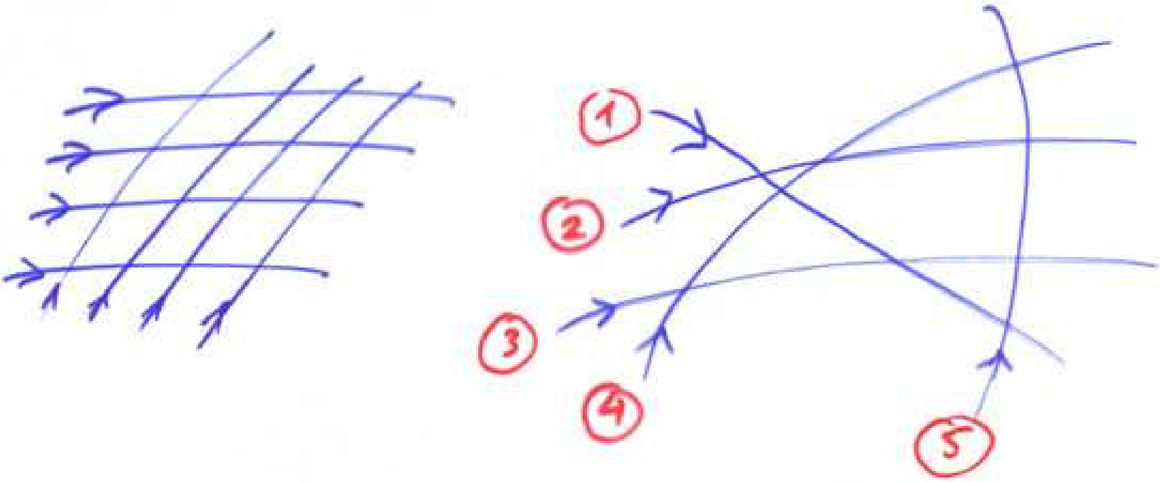
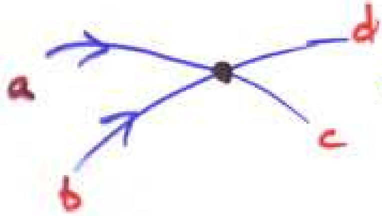
1. Introduction to the Yang-Baxter equation
2. Algebraic structures, functional equations and their connections to TBA.
3. Difference property
4. Hidden 3D structures in 2D models.
5. Conclusion.

1. Introduction to the Yang-Baxter Eqns.

(2D lattice models of stat mech.)



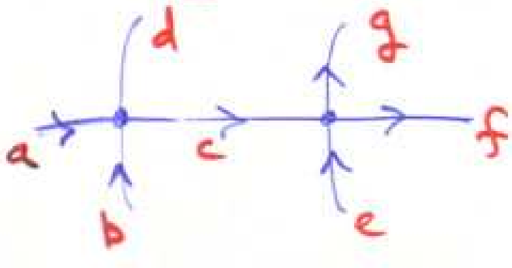
* Oriented 4-valent graph
* Edges carry "spin" variables $a, b, c \ldots \in \{1, \ldots, N\}$



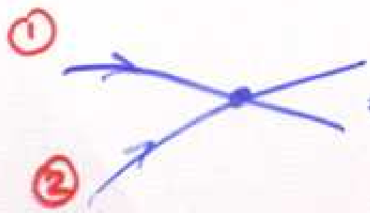
$= R^{cd}_{ab}$ (local Boltzmann weights)

$$Z = \sum_{(\text{spins})} \prod_{(\text{vertices})} R^{cd}_{ab} \quad \text{(partition function)}$$

Geometric multiplication (diagram rules)



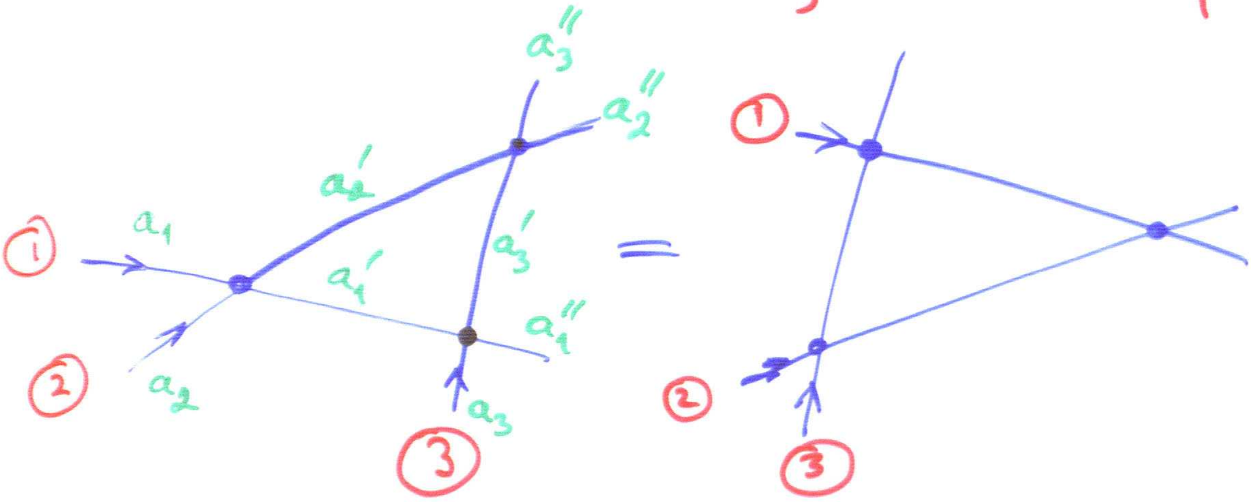
$= \sum_{c} R^{cd}_{ab} R^{fg}_{ce}$



$= R_{12}$

* summation over internal spins
* external spins are fixed
* R^{cd}_{ab} are different for different vertices.

Yang-Baxter Equ.



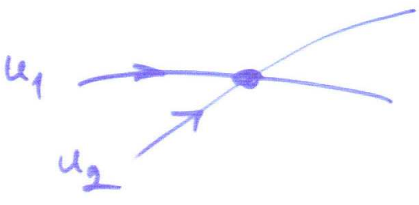
$$\sum_{a_1' a_2' a_3'} (R_{12})^{a_1' a_2'}_{a_1 a_2} \, (R_{13})^{a_1'' a_3'}_{a_1' a_3} \, (R_{23})^{a_2'' a_3''}_{a_2' a_3'} =$$

$$= \sum{}' \; - - - - - - - - - - -$$

$$R: \; V \otimes V \to V \otimes V, \quad V = \mathbb{C}^N$$

$$R_{12} R_{13} R_{23} = R_{23} R_{13} R_{12}$$ || $P(x \otimes y) = (y \otimes x)$.

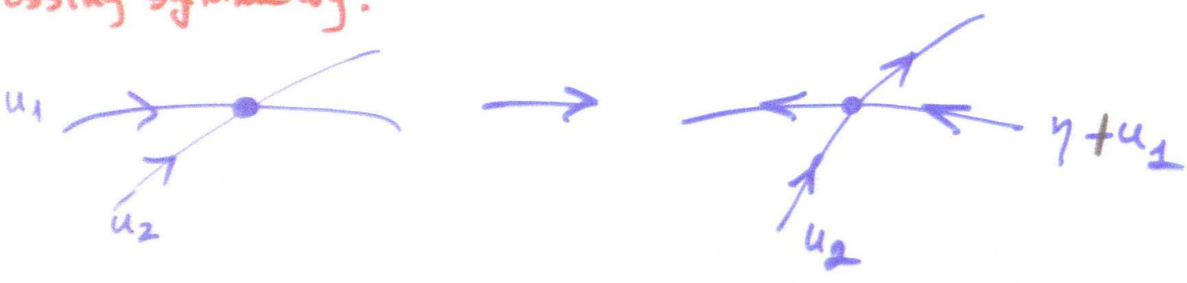
$$R_{12} = R(u_1 - u_2)$$

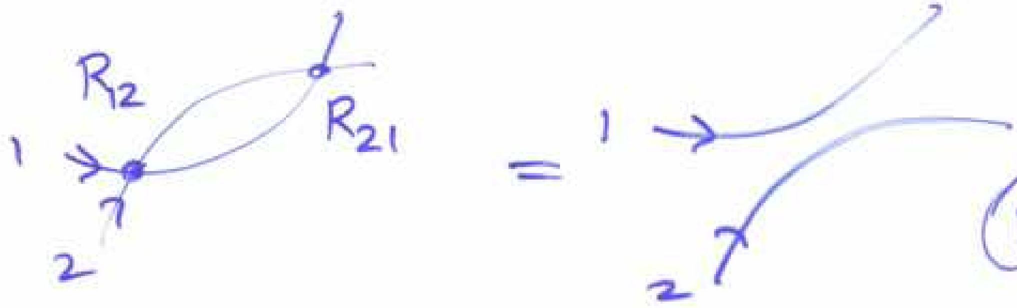


"difference property"

$$R(u) = (C \otimes 1)\left(P\, R(-\eta - u)\, P\right)^{t_1} (C \otimes 1)^{-1}$$

"crossing symmetry".



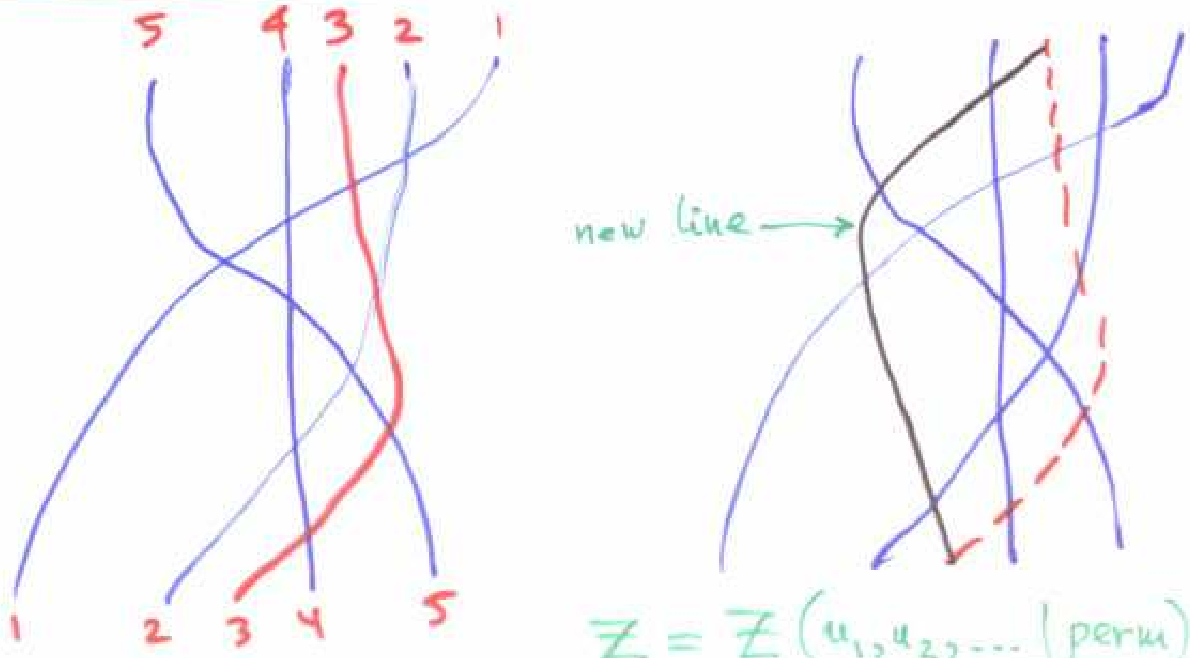


$(R_{12} R_{21} = 1)$

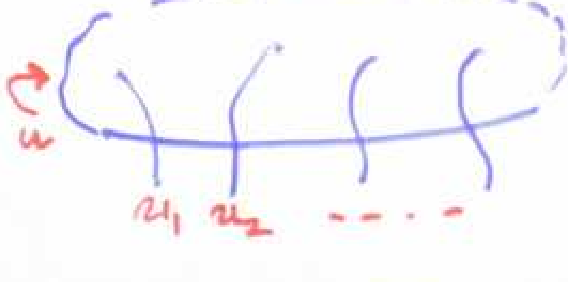
$R(u)\, P\, R(-u)\, P = 1$ "unitarity".

Z-invariance (Baxter 1979).

Partition function depends only on boundary data, but not on details of the lattice inside (i.e. on a permutation & rapidities)



$Z = Z(u_1, u_2, \ldots \mid \text{perm})$.



$= T(u)$

"transfer matrix"

$\left[T(u), T(v) \right] = 0, \quad \forall\, u, v$

"integrability".

(4) 2. Solutions of the Yang-Baxter Eq. (YBE)

YBE is an overdetermined system of algebraic eqns. (N^6 eqns. for $3N^4$ unknowns). We do not know a general solution even for $N=2$!

But there are recèpes, e.g., Quantum Groups

Consider quantum Kac-Moody algebra $\mathcal{A}=U_q(\widehat{sl_2})$

$x_0, x_1, y_0, y_1, h_0, h_1$ - (Cartan-Weyl generators)

$$\begin{cases} [h_i, h_j]=0, \ [h_i, x_j]=-a_{ij}x_j, \ [h_i\, y_j]=a_{ij}y_j \\ [x_i, y_j]=\delta_{ij}\dfrac{q^{h_i}-q^{-h_i}}{q-q^{-1}}, \quad \|a_{ij}\|=\begin{pmatrix}2 & -2\\ -2 & 2\end{pmatrix} \\ + \text{ 4-th order Serre relations} \end{cases}$$

$K=h_0+h_1=0$ (central element)

$\Delta: \quad \mathcal{A}\to\mathcal{A}\otimes\mathcal{A}$ (co-multiplication)

$$\Delta(x_i)=x_i\otimes 1+q^{-h_i}\otimes x_i$$

$$\Delta(y_i)=y_i\otimes q^{h_i}+1\otimes y_i$$

$$\Delta(h_i)=h_i\otimes 1+h_i\otimes 1$$

(5) There exist a "universal R-matrix" (Drinfeld).

$$\mathcal{R} \subset B_+ \otimes B_- ; \quad B_-: \{h_{0,1}, x_0, x_1\}$$
$$B_+: \{h_{0,1}, y_0, y_1\}$$

$$\Delta' \mathcal{R} = \mathcal{R} \Delta ; \quad \Delta' = P \circ \Delta$$

$$\mathcal{R}^{12} \mathcal{R}^{13} \mathcal{R}^{23} = \mathcal{R}^{23} \mathcal{R}^{13} \mathcal{R}^{12} \quad \text{(YBE)}$$

$\mathcal{R}^{12} = \mathcal{R} \otimes 1$, etc.

"Explicit expression for $\mathcal{R}$"

$$\mathcal{R} = q^{-\frac{h_0 \otimes h_0}{2}} \Big(1 + (q - q^{-1})(y_0 \otimes x_0 + y_1 \otimes x_1) +$$
$$+ \text{infinite series in } y_i \otimes 1, 1 \otimes x_i \Big)$$

can be evaluated for particular repr. of $U_q(\hat{sl}_2)$.

Representations of $U_q(\hat{sl}_2)$

* YBE is written in $\mathcal{B}_+ \otimes \mathcal{A} \otimes \mathcal{B}_-$

* $U_q(\hat{sl}_2) \xrightarrow{\text{"evaluation"}} U_q(sl_2)$

* $B_\pm(U_q(sl_2)) \longrightarrow H_q$ (deformed Heisenb… q-oscillators.

(ii) Evaluation representations $U_q(\hat{sl}_2) \to U_q(sl_2)$

$U_q(sl_2)$: $[H,E]=2E$, $[H,F]=-2F$,

$$[E,F] = \frac{q^H - q^{-H}}{q - q^{-1}}$$

$x_0 \to \lambda^{-1} F q^{-H/2}$, $y_0 \to \lambda q^{H/2} E$, $h_0 \to H$

$x_1 \to \lambda^{-1} E q^{H/2}$, $y_1 \to \lambda q^{-H/2} F$, $h_1 \to -H$

$\lambda = e^{iu}$ - multiplicative spectral parameter.

Representations of $U_q(sl_2)$

① π_j, $j = 0, 1/2, 1, \dots,$ $(2j+1)$ dimensional reps.

② π_j^+, $j \in \mathbb{C}$, infinite dimensinal highest weight reps.

③ π_{cyclic}, $q^n = 1$, ~~[illegible]~~ n-dimensional reps. without highest weight.

(b) q-oscillator reps. $B_+(U_q(\hat{sl}_2)) \to H_q$

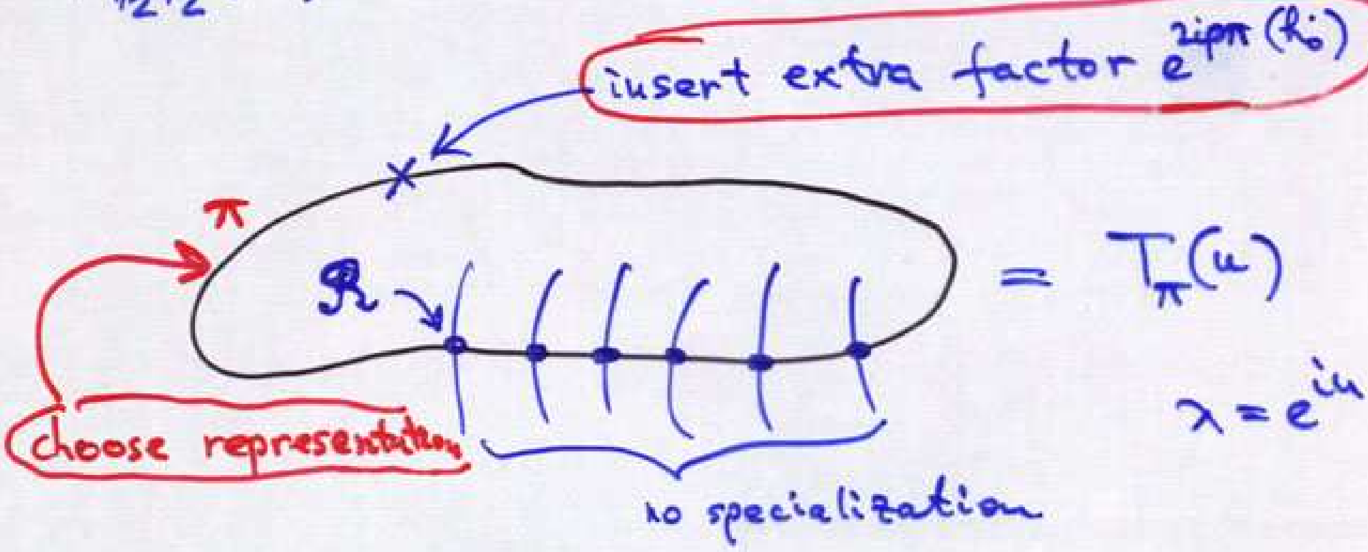
$\rho_\pm$: $h_0 = -h_1 \to \pm\mathcal{H}$, $y_0 \to \lambda\,\mathcal{E}_\pm$, $y_1 \to \lambda\,\mathcal{E}_\mp$

$$\boxed{H_q:\ q\,\mathcal{E}_+\mathcal{E}_- - q^{-1}\mathcal{E}_-\mathcal{E}_+ = \frac{1}{q-q^{-1}},\quad [\mathcal{H}, \mathcal{E}_\pm] = \pm 2\,\mathcal{E}_\pm}$$

Functional relations for T-operators

$$R_{jj'}(\lambda/\mu) = (\pi_j(\lambda) \otimes \pi_{j'}(\mu))\, \mathcal{R}$$

$R_{1/2\,1/2}(\lambda)$ – R-matrix of the 6-vertex model.



$= T_\pi(u)$

$\lambda = e^{iu}$

"Universal" T-operators are elements of

$\left(\mathcal{B}_- \otimes \mathcal{B}_- \otimes \mathcal{B}_- \cdots \otimes \mathcal{B}_-\right\}$; $\mathcal{B}_- = \mathcal{B}_-(U_q(\widehat{sl}_2))$

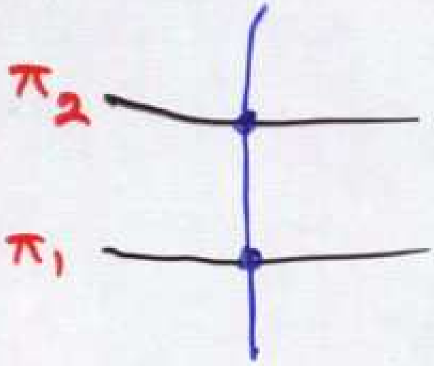
$T_j(u)$ – when $\pi = \pi_j$, $(2j+1)$-dim

$j = 0, 1/2, 1, \ldots$

$T_j^+(u)$ when $\pi = \pi_j^+$, infinite-dim h.w. reps.

$Q_\pm(u)$ when $\pi = \rho_\pm$, q-oscillators.

Algebra of T-operators.



$$= \underbrace{\begin{pmatrix} R^{(1)} & * \\ 0 & R^{(2)} \end{pmatrix}}_{\dim \pi_1 \cdot \dim \pi_2}$$

for special shifts of the spectral parameters.

$$T_j^+(u) = Q_+(u+(2j+1)\eta)\, Q_-(u-(2j+1)\eta)$$

$$q = e^{2i\eta}$$

$$T_j(u) = T_j^+(u) - T^+_{-j-1}(u), \text{ when } (2j+1) \in \mathbb{Z}_+$$

$$T_j(u) = Q_+(u+(2j+1)\eta)\, Q_-(u-(2j+1)\eta) - Q_+(u-(2j+1)\eta)\, Q_-(u+(2j+1)\eta)$$

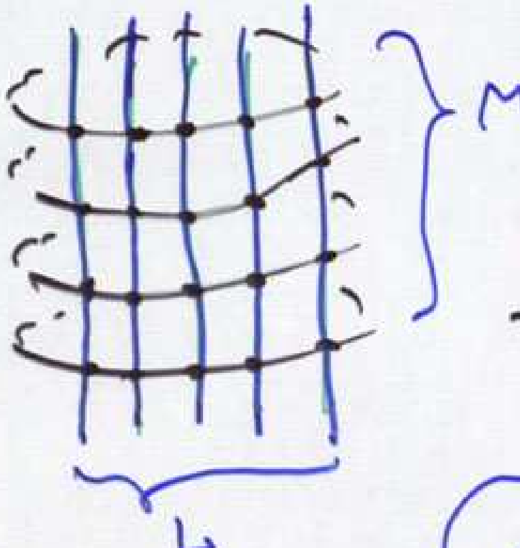
$$T_0(u) = 1,$$

$$\boxed{T_{1/2}(u)\, Q_\pm(u) = Q_\pm(u+\eta) + Q_\pm(u-\eta)} \quad \text{TQ-equ.}$$

$$T_j(u+\eta)\,T_j(u-\eta) = 1 + T_{j-\frac{1}{2}}(u)\,T_{j+\frac{1}{2}}(u)$$

$$Y_j(u+\eta)\,Y_j(u-\eta) = \left(1+Y_{j-\frac{1}{2}}(u)\right)\left(1+Y_{j+\frac{1}{2}}(u)\right)$$

$$Y_j(u) = T_{j-\frac{1}{2}}(u)\,T_{j+\frac{1}{2}}(u)\,, \quad j=\tfrac{1}{2}, 1, \ldots$$



* Equations are exact for finite L
* universal for lattice/field th.

$$\mathrm{Tr}\left(T_L\right)^M = \mathrm{Tr}\left(\widetilde{T}_M\right)^L$$

finite volume side

TBA side

* TBA strings correspond to reps. of quantum group.

For $L \to \infty$ $\quad Y_j(u) \simeq \left(f_j(u)\right)^L$

$\log Y_j(u) = L \log f_j(u) +$ integral term that vanish when $L \to \infty$.

Note $\quad \widetilde{T}_M{}^L \neq e^{-L\widetilde{H}}$

For TBA with $\mathrm{Tr}\left(e^{-L\widetilde{H}}\right)$ the analytic properties could be extremely complicated.

About the difference property
(and its absence).

1. Cyclic representation at roots of unity, $q^n = 1$.

2. Quantum systems on a classical background.

3. Hidden 3d structure

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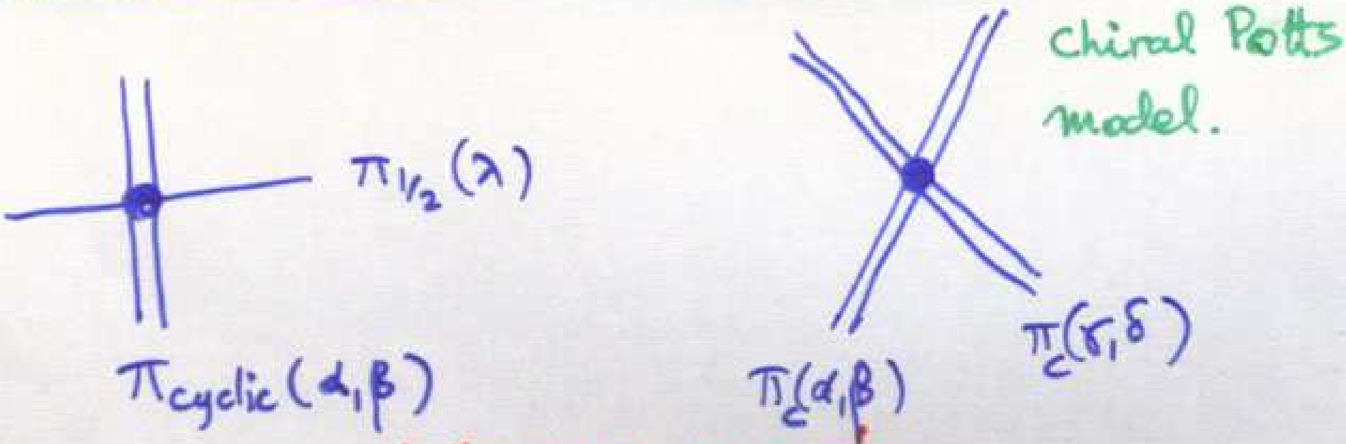
1. <u>Cyclic reps. of $U_q(sl_2)$.</u>

for $q^n=1$ the n-th powers E^n, F^n are central elements.

$\boxed{\text{$\pi_{cyclic}$ contains two spectral parameters which lie on a curve of the genus $(n-1)^2$}}$

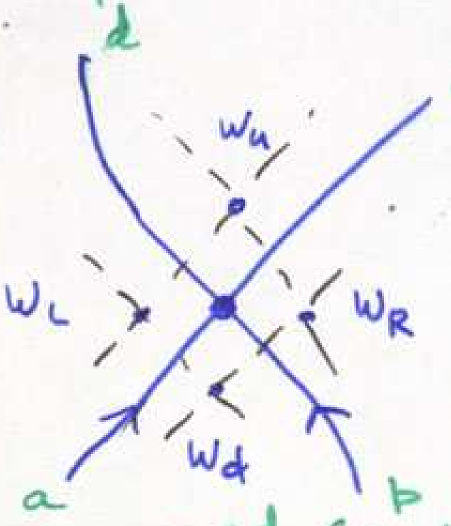


No difference property.

Quantum systems on a classical background.

* fluctuations around classical solutions.
* parameters of the quantum model depend on the classical solution of the equations of motion.

(Hirota eqns).



$$R_{ab}^{cd}(w_u/w_d)$$

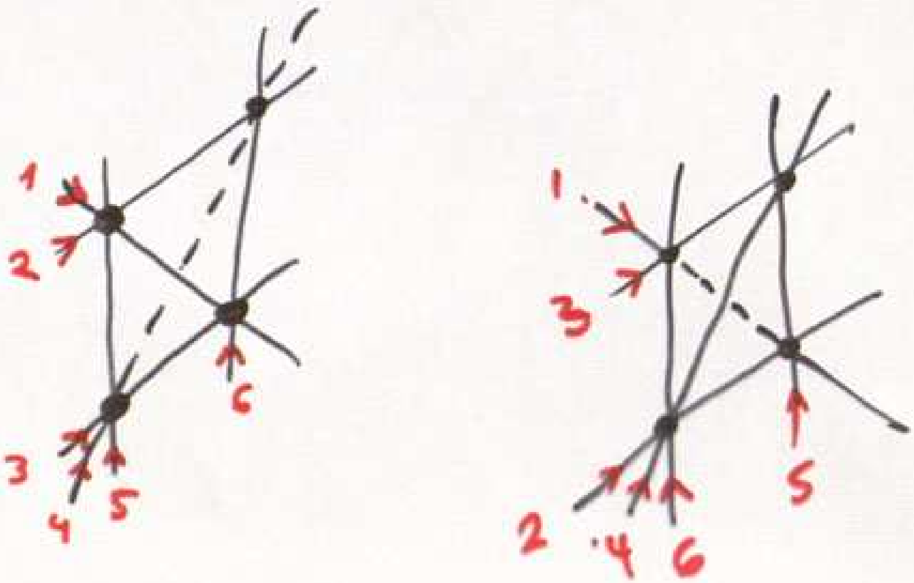
$$\frac{w_u}{w_d} = \frac{x\,w_L + w_R}{w_L + x\,w_R}$$

Faddeev & Volkov
Bobenko & Pinkal
VB, Bobenko & Reshetikhin

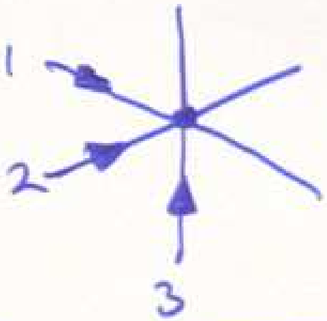
* Solve classical eqn. of motion
* Calculate Boltzmann weights
* For a trivial constant classical solution one obtains the chiral Potts model again (VB 2008).
* the same Hirota eqn. for central charges arise for SU(2|2) R-matrix (in Beisert, Arutyunov & Frolov)

3D structure

1. Zamolodchikov tetrahedron equation
(3D analog of the Yang-Baxter eqn).



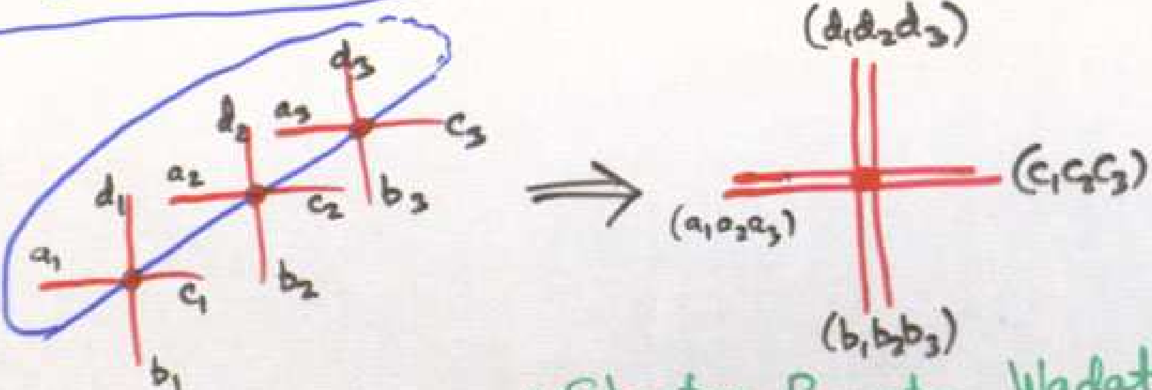
$$R_{345}\, R_{125}\, R_{136}\, R_{246} = R_{246}\, R_{136}\, R_{125}\, R_{345}$$



$= R_{123}$

* VB & Baxter (1993) (Chiral Potts)
* VB & Sergeev (2006) $U_q(\widehat{gl}_n)$

Spectral parameters & moduli are mixed under 3D rotation.



* Shastry R-matr. Wadati

Conclusion

* What is a meaning of Z - invariance for the action functional?

* functional equations lead to the TBA, strings corresponds to reps of quantum group. (No string conjectures required).
* For $q^N = 1$ there is only a finite number of string types.

* How to include $SU(2|2)$ R-matrix into the general quantum group scheme? Are there missing spectral parameters?

* Are there new solutions of the tetrahedron equation behind the AdS/CFT?

The following bibliography comments have been appended to the original scanned document on 06 July 2009.

1. Z-invariance
   (a) the primary reference is Baxter [1].
   (b) for applications to link invariants see Jones [2].
   (c) for applications to one-point functions in the chiral Potts model see Baxter [3]
   (d) for connections with rhombic tilings and circle patterns see [4]
2. Universal R-matrix
   (a) the primary reference is Drinfeld [5]
   (b) for an expression in terms of infinite products see Khoroshkin and Tolstoy [6]
   (c) for an example of evaluation for $U_q(\widehat{sl_2})$ see [7]
3. String hypotheses in TBA
   (a) examples of cut-offs to the number of string types in the context of the RSOS model for $U_q(\widehat{sl_2})$ [8] and $U_q(\widehat{sl_n})$ [9]
   (b) a really *wild* string hypothesis connecting $U_q(A_2^{(2)})$ with $U_q(\widehat{E_8})$, [10].
4. Functional relations
   (a) Q-operator, see Appendix C of [11], which (along with other four appendices and, of course, the main text of the paper) contains a wealth of information on the subject. Highly recommended.
   (b) details of the derivation of the functional relations using Q-operators and their connection to the q-oscillators are given in [12]. For some other algebras see [13] and [14] and references therein. Alternative algebraic construction for Q-operators in the context of cyclic representations is given in [15]
   (c) integral equations for excited states [16]
   (d) connection of functional relations to the TBA is discussed in [17] and [18]
5. The (absence of) difference property
   (a) the primary example of a model without the difference property is the chiral Potts model, see [19].
   (b) for a connection of the chiral Potts model to quantum groups and cyclic representations see [15]
   (c) integrable quantum spin systems on an (integrable) classical background [20, 21].
   (d) Hidden 3D structure
      i. primary references for the tetrahedron equation are [22, 23].
      ii. hidden 3D structure of 2D models — see [24] and [25].
      iii. chiral Potts model as a two-layer 3D model [24].
      iv. meaning of the tetrahedron equation in discrete differential geometry [26]

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