# Integrability, ....

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# Classical (Hamiltonian) Mechanics

- $\mathbb{R}^{2n}$, $(p_1 \cdots p_n, q_1 \cdots q_n)$
- $C(\mathbb{R}^{2n})$,

$$\{F, G\} = \sum_{i=1}^{n} \left( \frac{\partial F}{\partial p_i} \frac{\partial G}{\partial q_i} - \frac{\partial F}{\partial q_i} \frac{\partial G}{\partial p_i} \right)$$

- $H \in C(\mathbb{R}^{2n}) \rightsquigarrow \dot{x}_i = \{H, x_i\}$
- <u>integrable</u> : $\exists\, I_1, \ldots, I_n \in C(\mathbb{R}^{2n})$

  (i) independent

  (ii) $\{I_i, I_j\} = 0$

  (iii) $\{I_i, H\} = 0$

$H = H(I_1, \ldots, I_n)$

$(dH \wedge dI_1 \wedge \cdots \wedge dI_n \equiv 0)$

(2)

- $M_c = \{x \in \mathbb{R}^{2n} \mid I_i(x) = c_i\}$
  has affine coord. system $\left(\begin{array}{c}\text{charts} \leftrightarrow \\ \leftrightarrow \text{aff.}\end{array}\right)$
  $\phi_1, \dots \phi_n$

  (i) $\phi_i(t) = \omega_i t + \phi_i(0)$
  (ii) $(I_i, \phi_i)$ = action / angle
  $\{\phi_i, \phi_j\} = 0$, $\{I_i, \phi_j\} = \delta_{ij}$

- Fundamental ingradient $= I_1, \dots, I_n$

- Questions:
  - classical spectrum (values of H)
  - how to "solve"
    $\left(\begin{array}{l}\text{how to find} \\ (p_i, q_i) \to (I_i, \phi_i)?\end{array}\right)$

- Lax pair (P. Lax, ....)

$$L, M : \mathbb{R}^{2n} \longrightarrow \mathrm{End}(V)$$

$$\left( \dot{x}_i = \{H, x_i\} \right) \Longleftrightarrow \left( \frac{dL}{dt} = [L, M] \right)$$

integr. of motion = spectr. inv. of $L$

$$\det(\lambda - L) = \sum_i \lambda^i E_i \,, \quad I_i = \mathrm{tr}(L^i), \dots$$

commutativity of integrals?

in some cases give solutions.

- Lax + Hamiltonian structures:

$$\{L_{ij}, L_{kl}\} = ?$$

## Classical r-matrix (Sklyanin): ④

- $r : V \otimes V \to V \otimes V$

$$[r_{12}, r_{13}] + [r_{12}, r_{23}] + [r_{13}, r_{23}] = 0$$

$$r_{12} = r \otimes I, \quad r_{23} = I \otimes r, \ldots$$

- Assume:

$$\{L \overset{\otimes}{,} L\} = [r, L \otimes L] \quad \left(\begin{array}{c}\text{act in} \\ V \otimes V\end{array}\right)$$

$$\{L \overset{\otimes}{,} L\}_{ij,kl} = \{L_{ij}, L_{kl}\}$$

Then:

$$\{tr(L^i), tr(L^j)\} = 0, \quad \forall i,j$$

commuting integrals

"r-matrix method"

- find $r$
- Consider $\{L \overset{\otimes}{,} L\} = [r, L \otimes L]$
- fix a Hamiltonian

$$H = f(L) = f(ULU^{-1})$$

- Equations of motion are Lax can be solved by factorization method

* $(G, r)$ Poisson Lie group
[Factorizable : $r_{12} + r_{21} = I_{12}$]

* Such structures for simple Lie algebras and for loop alg. are classified ( Belavin, Drinfeld)
Superalgebras (Karaali ) ...

# Example

$V = \mathbb{C}^2$ "marked" by $\lambda \in \mathbb{C}$

$$r = \frac{\sum_{a=1}^{3} \sigma^a \otimes \sigma^a}{\lambda - \mu}, \quad \lambda, \mu = \text{"marks"}$$

- $\left[ r_{12}(\lambda_1 - \lambda_2), r_{13}(\lambda_1 - \lambda_3) \right] + \cdots = 0$

- $r_{12}(\lambda_1 - \lambda_2 + i0) + r_{21}(\lambda_2 - \lambda_1 - i0) =$
  $= -2\pi i \, \delta(\lambda_1 - \lambda_2) \sum_{a=1}^{3} \sigma^a \otimes \sigma^a$

- $\{ L_1(\lambda) \otimes L_2(\mu) \} = \left[ r_{12}(\lambda - \mu), L(\lambda) \otimes L(\mu) \right]$
  Poisson brackets

- $L(\lambda) = \lambda + \sum_{a=1}^{3} \sigma^a S^a$
  $\{ S^a, S^b \} = \varepsilon^{abc} S^c$

# SO(3) spin chain

$(S^2)^{\times N}$ ; $S_n^a$ , $\vec{S}_n^2 = \Delta_n^2$

$a = 1,2,3$ ; $n = 1,2,\dots,N$

$$\{S_n^a, S_m^b\} = \delta_{nm}\, \varepsilon^{abc} S_n^c$$

$$L_n(\lambda) = \lambda + \sum_{a=1}^{3} \sigma^a S_n^a$$

$$T(\lambda) = L_N(\lambda - \lambda_N) \cdots L_1(\lambda - \lambda_1)$$

$$t(\lambda) = \mathrm{tr}(T(\lambda))$$

cl. YBE. $\Rightarrow$ $\{t(\lambda), t(\mu)\} = 0$

$$t(\lambda) = 2\lambda^N + \sum_{i=0}^{N-1} \lambda^i t_i$$

Any $H = f(t_0, \dots, t_{N-1})$ is integrable .... | $SL_2(\mathbb{R})$ ....

## Special case

$\Delta_n = \Delta$, $\quad \lambda_{2n} = a$, $\lambda_{2n-1} = -a$

$$t(\lambda) = \mathrm{tr}\left(L_{2N}(\lambda+a) L_{2N-1}(\lambda-a) \dots L_1(\lambda-a)\right)$$

Take

$$H = \ln\left(t(a+\Delta)\, t(a-\Delta)\, t(-a+\Delta)\, t(-a-\Delta)\right)$$

Claim: as $\Delta \to 0$

- $S^a_{2n} = \Delta\, S^a(2n\Delta)$, $S^a_{2n-1} = \Delta\, T^a((2n+1)\Delta)$

$$\boxed{\{S^a(x), S^b(y)\} = \varepsilon^{abc} S^c(x)\,\delta(x-y)}$$

same for $T$, $\{S, T\} = 0$

- Equations of motion $\to$

$$\boxed{\begin{aligned} \partial_t \vec{S} &= \partial_x \vec{S} + \frac{1}{a}\, \vec{S} \times \vec{T} \\ \partial_t \vec{T} &= +\partial_x \vec{T} - \frac{1}{a}\, \vec{S} \times \vec{T} \end{aligned}}$$

$$\boxed{H = P_S + P_T + \frac{1}{a} \int \vec{S}\,\vec{T}\, dx}$$

Another Ham. system with the same equations of motions

$$\{S^a(x), S^b(y)\} = \varepsilon^{abc} S^c(x)\delta(x-y)$$

$$\{S^a(x), T^b(y)\} = \delta^{ab}\delta'(x-y)$$

$$\{T^a(x), T^b(y)\} = \varepsilon^{abc} T^c(x)\delta(x-y)$$

$$H = \int \vec{S}(x)\vec{T}(x)\,dx$$

$$\left( \mathcal{L} = \int tr(\partial_\mu g\, g^{-1} \partial_\mu g\, g^{-1})\,dx, \right.$$

$$\vec{S} = tr(\vec{\sigma}\,\partial_+ g\, g^{-1}), \quad \vec{T} = \cdots \partial_- \cdots$$

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one more

$$\vec{S} = \psi_+^* \vec{\sigma} \psi_+ \quad \vec{T} = \psi_-^* \vec{\sigma} \psi_-$$

$$\psi_{\alpha,i} \qquad \alpha = 1,2 \qquad i = 1,\dots, M$$

$$H = \int \left( \psi_+^* \partial_x \psi_- + \psi_-^* \partial_x \psi_+ + \frac{1}{a}\vec{S}\vec{T}\right)$$

$M$ = any

# Quantization

Fncns on the phase space with $\{\cdot,\cdot\}$ — Assoc. algebra $\underset{\sim}{A}$ of quant. obs.

$f * g = fg + \frac{i\hbar}{2}\{f,g\} + \dots$

Hamiltonian $H$ = fncn on the phase space — quantum Hamiltonian $H \in A$ acting on a repr. space of $A$

Integrable system $I_1, \dots, I_n$ $\{I_i, I_j\} = 0$ — maximal commutative subalg. of $A$ represented in a repr. space

How to: • quantize

• quantize an int. system ?

# Quantization of Lax oper.

(Faddeev, Sklyanin)

$r$ — $R = 1 + \hbar r + \dots$

$[r_{12}, r_{13}] + \dots = 0$ — $R_{12} R_{13} R_{23} = R_{23} R_{13} R_{12}$

$\{L \overset{\otimes}{,} L\} = [r, L \otimes L]$ — $R_{12} L_1 L_2 = L_2 L_1 R_{12}$

$I_i = tr(L^i)$ — $\hat{I}_i = \text{"qtr"}(L^i)$

The result: $\hat{I}_i$ acting in the quantum space of states

Next problem: $\mathrm{Spec}(\hat{I}_i) = ?$

## Example:

$$r(\lambda) = \frac{\sum_a \sigma^a \otimes \sigma^a}{\lambda} \quad — \quad R(\lambda) = \lambda + P$$

$$P(x \otimes y) = y \otimes x$$

act in $\mathbb{C}^2 \otimes \mathbb{C}^2$

$$R_{12}(\lambda-\mu) L_1(\lambda) L_2(\mu) = L_2(\mu) L_1(\lambda) R_{12}(\lambda-\mu)$$

$$L(\lambda) = \lambda + \sum_{a=1}^{3} \sigma^a \otimes S^a$$

$$[S^a, S^b] = i \varepsilon^{abc} S^c$$

$$T(\lambda) = L_{(N)}(\lambda - \lambda_N) \cdots L_{(1)}(\lambda - \lambda_1)$$

$$t(\lambda) = \mathrm{tr}(T(\lambda))$$

$$T = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

$$[t(\lambda), t(\mu)] = 0$$

## Spectrum:

$$T(\lambda) = \begin{pmatrix} A(\lambda) & B(\lambda) \\ C(\lambda) & D(\lambda) \end{pmatrix}$$

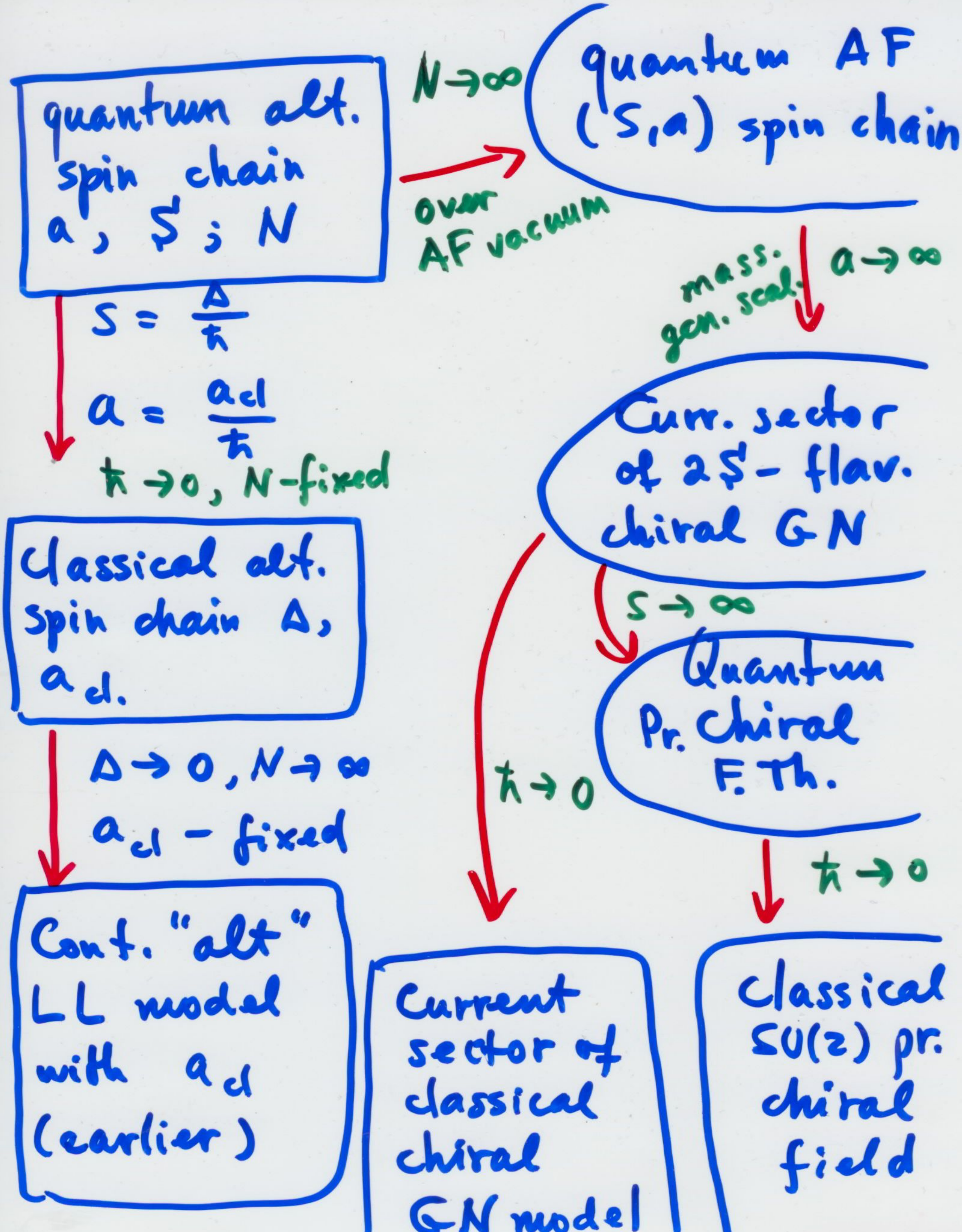
| | $SU(2)$ | $SL_2(\mathbb{R})$ |
|---|---|---|
| irr. unitary repr. | $V^j$ f.d. | • $U_t$ – princ. unitary (no h.w.v. / no l.w.v.) <br> • $D^j_\pm$ – discrete series <br> • (exceptional) |


For $V^j$, $D^j_\pm$ alg. Bethe ansatz:

$$\Psi(v_1, \dots, v_n) = B(v_1) \cdots B(v_n)\,\Omega$$

$$t(\lambda)\,Q(\lambda) = a(\lambda)\,Q(\lambda+1) + \delta(\lambda)\,Q(\lambda-1)$$

$$Q(\lambda) = (\lambda - v_1) \cdots (\lambda - v_n) .$$



quantum alt. spin chain
$a$, $\tilde{S}$ ; $N$
$N \to \infty$
over AF vacuum
quantum AF
$(S, a)$ spin chain
mass. gen. scal.
$a \to \infty$
$S = \frac{\Delta}{\hbar}$
$a = \frac{a_{cl}}{\hbar}$
$\hbar \to 0$, $N$-fixed
Curr. sector of $2\tilde{S}$- flav. chiral GN
$S \to \infty$
Quantum Pr. Chiral F. Th.
Classical alt. spin chain $\Delta$, $a_{cl}$.
$\Delta \to 0$, $N \to \infty$
$a_{cl}$ - fixed
$\hbar \to 0$
$\hbar \to 0$
Cont. "alt" LL model with $a_{cl}$ (earlier)
Current sector of classical chiral GN model
Classical SU(2) pr. chiral field


F&NR

For $U_t$ : Sklyanin's separation of variables

$$t(\lambda)\, Q(\lambda) = a(\lambda)\, Q(\lambda+1) + b(\lambda)\, Q(\lambda-1)$$

but $Q(\lambda)$ is in a certain class of fncns and has $\infty$ many zeroes.

The thermodynamical (continuum, scaling ...) limit:

$N \to \infty$ , $\dots \to \dots$

$\mathcal{H}_N = (\mathbb{C}^2)^{\otimes N}$ ( or $(V^j)^{\otimes N}$, or ...)

$\Omega_N \in \mathcal{H}_N$ , ... $\infty$-dim var. of vacua

$\mathcal{O}\Omega_N \to \mathcal{H}_\infty$