The S-Matrix Reloaded: Twistors, Unitarity, Gauge Theories and Gravity

Potsdam 2006: Integrability in Gauge and String Theory
Zvi Bern, UCLA

The past two years have seen a significant advance in our ability to compute scattering amplitudes.

- The call of the LHC: multi-parton scattering at loop level.
- Can we resum (planar) $N = 4$ super-Yang-Mills theory?
- The structure of perturbative quantum gravity. Reexamine standard wisdom on quantum gravity.
LHC Physics

The LHC will start operations in 2007.

We will have lots of multi-particle processes. Want reliable predictions.
Example: Susy Search

Early ATLAS TDR studies using PYTHIA overly optimistic.

- **ALPGEN** is based on LO matrix elements and much better at modeling hard jets.

- What will disagreement between ALPGEN and data mean? Hard to tell. Need NLO.

Such a calculation is well beyond anything that has been done using Feynman diagrams.

We need $pp \rightarrow Z + 4$ jets at NLO
State-of-the-Art NLO QCD

Five point is still state-of-the art for QCD cross-sections:

Typical examples:

\[ pp \rightarrow W, Z + 2 \text{ jets} \]

\[ pp \rightarrow \bar{b}bH \text{ or } pp \rightarrow \bar{t}tH \]

Brute force calculations give GB expressions – numerical stability?
Amusing numbers: 6g: 10,860 diagrams, 7g: 168,925 diagrams
Much worse difficulty: integral reduction generates nasty dets.

\[ \frac{1}{\det(k_i \cdot k_j)^n} \]

“Grim” determinant
Experimenters to theorists: **“Please calculate the following at NLO”**

<table>
<thead>
<tr>
<th>Single boson</th>
<th>Diboson</th>
<th>Triboson</th>
<th>Heavy flavour</th>
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Theorists to experimenters: **“In your dreams”**

A key theoretical problem for LHC is NLO
### More Realistic NLO Wishlist

*Les Houches 2005*

<table>
<thead>
<tr>
<th>process ( (V \in {Z, W, \gamma}) )</th>
<th>background to</th>
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<tbody>
<tr>
<td>1. ( pp \rightarrow VV ) jet</td>
<td>( t\bar{t}H ), new physics</td>
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<tr>
<td>2. ( pp \rightarrow H + 2 ) jets</td>
<td>( H ) production by vector boson fusion (VBF)</td>
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<tr>
<td>3. ( pp \rightarrow t\bar{t}b\bar{b} )</td>
<td>( t\bar{t}H )</td>
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<tr>
<td>4. ( pp \rightarrow t\bar{t} + 2 ) jets</td>
<td>( t\bar{t}H )</td>
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<tr>
<td>5. ( pp \rightarrow VV b\bar{b} )</td>
<td>VBF( \rightarrow H \rightarrow VV ), ( t\bar{t}H ), new physics</td>
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<tr>
<td>6. ( pp \rightarrow VV + 2 ) jets</td>
<td>VBF( \rightarrow H \rightarrow VV )</td>
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<tr>
<td>7. ( pp \rightarrow V + 3 ) jets</td>
<td>various new physics signatures</td>
</tr>
<tr>
<td>8. ( pp \rightarrow VVV )</td>
<td>SUSY trilepton</td>
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**Bold action required!**
Consider an integral

\[ \int \frac{d^{4-2\epsilon} \ell}{(2\pi)^{4-\epsilon}} \frac{\ell^\mu \ell^\nu \ell^\rho \ell^\lambda}{\ell^2 (\ell - k_1)^2 (\ell - k_1 - k_2)^2 (\ell + k_4)^2} \]

Evaluate this integral via Passarino-Veltman reduction. Result is ...
Result of performing the integration

Numerical stability is a key issue. Clearly, there should be a better way.
Why are Feynman diagrams clumsy for high loop or multiplicity processes?

- Vertices and propagators involve gauge-dependent off-shell states. Origin of the complexity.

\[ \cdots + \cdots + \cdots \]

\[ p^2 \neq 0 \]

- To get at root cause of the trouble we must rewrite perturbative quantum field theory.

- All steps should be in terms of gauge invariant on-shell states. \[ p^2 = m^2 \] On shell formalism.
- Radical rewrite of gauge theory needed.
“One of the most remarkable discoveries in elementary particle physics has been that of the existence of the complex plane.”

*J. Schwinger in “Particles, Sources and Fields” Vol 1*
With on-shell formalisms we can exploit analytic properties

- **Curiously, a practical on-shell formalism was constructed at loop level prior to tree level: unitarity method.** Bern, Dixon, Dunbar, Kosower (1994)

- **Solution at tree-level had to await Witten’s twistor inspiration.** (2004)
  - -- MHV vertices Cachazo, Svrcek Witten; Brandhuber, Spence, Travaglini
  - -- On-shell recursion Britto, Cachazo, Feng, Witten

- **Combining unitarity method with on-shell recursion gives loop-level on-shell bootstrap.** Berger, Bern, Dixon, Forde, Kosower (2006)
Spinors and Twistors

Spinor helicity for gluon polarizations in QCD:

\[ \varepsilon_\mu^+(k; \mathbf{q}) = \frac{\langle \mathbf{q}^- | \gamma_\mu | k^- \rangle}{\sqrt{2} \langle \mathbf{q} \mathbf{k} \rangle}, \quad \varepsilon_\mu^-(k, \mathbf{q}) = \frac{\langle \mathbf{q}^+ | \gamma_\mu | k^+ \rangle}{\sqrt{2} [k \mathbf{q}]} \]

\[ \varepsilon^{ab}_\mu \lambda_j^a \lambda_{lb} \longleftrightarrow \langle j \mathbf{l} \rangle = \langle k_{j-} | k_{l+} \rangle = \sqrt{2 k_j \cdot k_l} e^{i \phi} = \frac{1}{2} \bar{u}(k_j)(1 + \gamma_5)u(k_l) \]

\[ \varepsilon_{\dot{a} \dot{b}} \tilde{\lambda}_j^{\dot{a}} \tilde{\lambda}_l^{\dot{b}} \longleftrightarrow [j \mathbf{l}] = \langle k_{j+} | k_{l-} \rangle = -\sqrt{2 k_j \cdot k_l} e^{-i \phi} = \frac{1}{2} \bar{u}(k_j)(1 - \gamma_5)u(k_l) \]

Penrose twistor transform:

\[ \tilde{A}(\lambda_i, \mu_i) = \int \prod_i \frac{d^2 \tilde{\lambda}_i}{(2\pi)^2} \exp \left( \sum_j \mu_j \tilde{\lambda}_j \tilde{\lambda}_{\dot{j}} \right) A(\lambda_i, \tilde{\lambda}_i) \]

Early work from Nair

Witten’s remarkable twistor-space link:

Witten; Roiban, Spradlin and Volovich

QCD scattering amplitudes \hspace{1cm} \longleftrightarrow \hspace{1cm} Topological String Theory
Witten conjectured that in twistor–space gauge theory amplitudes have delta-function support on curves of degree:

\[ d = q - 1 + L, \quad q = \# \text{ negative helicities}, \quad L = \# \text{ loops}, \]

**Structures imply an amazing simplicity in the scattering amplitudes.** Amplitudes are much much simpler than anyone imagined.

- Connected picture
- Disconnected picture

Witten
Roiban, Spradlin and Volovich
Cachazo, Svrcek and Witten
Gukov, Motl and Neitzke
Bena Bern and Kosower
MHV Amplitudes

At tree level Parke and Taylor conjectured a very simple form for $n$-gluon scattering.

$$A(1^-, 2^-, 3^+, \ldots, n^+) = i \frac{\langle 1 \ 2 \rangle^4}{\langle 1 \ 2 \rangle \langle 2 \ 3 \rangle \cdots \langle n \ 1 \rangle}$$

$$A(1^-, 2^-, 3^+, \ldots, n^+) = \sum_{\text{perms}} \text{Tr}[T^{a_1} T^{a_2} \cdots T^{a_n}] A(1^-, 2^-, 3^+, \ldots, n^+)$$

Proven by Berends and Giele

Amazingly, this simplicity continues to loops and to general helicities.

Bern, Dixon, Dunbar, Kosower
Cachazo, Svrcek, Witten; Bern, Dixon, Kosower
Brandhuber, Spence and Travaglini

These MHV amplitudes can be thought of as vertices for building new amplitudes.

Cachazo, Svrcek and Witten
Two-particle cut:

Three-particle cut:

Generalized unitarity:

As observed by Britto, Cachazo and Feng; quadruple cut freezes integral:

Coefficients of box integrals always easy.

Generalized cut interpreted as cut propagators not canceling.

Recent improvements for bubble and triangle contributions

Britto, Buchbinder, Cachazo and Feng; Britto, Feng and Mastrolia
New representations of tree amplitudes from IR consistency of one-loop amplitudes in $N = 4$ super-Yang-Mills theory. Using intuition from twistors and generalized unitarity: Bern, Del Duca, Dixon, Kosower; Roiban, Spradlin, Volovich

On-shell conditions maintained by shift. Proof relies on so little. Power comes from generality

- Cauchy’s theorem
- Basic field theory factorization properties
- Applies as well to massive theories
Shifted amplitude function of a complex parameter

\[ p_1^\mu(z) = p_1^\mu - \frac{z}{2} 1^- |\gamma^\mu| 2^- \]

\[ p_2^\mu(z) = p_2^\mu + \frac{z}{2} 1^- |\gamma^\mu| 2^- \]

Shift maintains on-shellness and momentum conservation

\[ A(z) = \sum \text{polylog terms} \quad \text{Use unitarity method} \]
\[ + \sum_i \text{Res}_{z_i} \frac{\text{Res}_{z_i}}{z - z_i} \quad \text{Use on-shell recursion} \]
\[ + \sum_i a_i z^i \quad \text{Use auxiliary on-shell recursion in another variable} \]

See David Kosower’s talk
### Numerical Results for $n$ Gluons

Choose specific points in phase-space – see hep-ph/0604195

#### Scalar loop contributions

<table>
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<tr>
<th>Helicity</th>
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</tr>
</tbody>
</table>

#### Naive diagram count

$+ 3,017,489$

other diagrams

Modest growth in complexity as number of legs increases

At 6 points these agree with numerical results of Ellis, Giele and Zanderighi
In 1974 ‘t Hooft proposed that we can solve QCD in planar (‘t Hooft) limit. This is too hard. $N = 4$ sYM is much more promising.

- Heuristically, we expect magical simplicity especially in planar limit with large ‘t Hooft coupling – dual to weakly coupled gravity in AdS space.

What we need

1. Intuition and bold guesses.
2. Sufficiently powerful methods for confirming and guiding guesses.
   (a) Unitarity method. Bern, Dixon, Dunbar and Kosower
   (b) A loop integration package: MB. Czakon
3. Faith and optimism.
Example: Twistor Space Hint

At one-loop the coefficients of all integral functions have beautiful twistor space interpretations.

Box integral

Twistor space support

Three negative helicities

Four negative helicities

The existence of such twistor structures implies loop-level simplicity. Supports notion that we should be able to evaluate amplitudes to all loop orders.

Bern, Dixon and Kosower
Britto, Cachazo and Feng
Consider one-loop in $N = 4$:

The basic $D$-dimensional two-particle sewing equation

$$\sum_{N=4 \text{ states}} A_4^{\text{tree}}(-\ell_1, 1, 2, \ell_2) \times A_4^{\text{tree}}(-\ell_2, 3, 4, \ell_1) = -\frac{stA_4^{\text{tree}}(1, 2, 3, 4)}{(\ell_1 - k_1)^2(\ell_2 - k_3)^2}$$

Applying this at one-loop gives

$$A_4^{1-\text{loop}}(1, 2, 3, 4) = -stA_4^{\text{tree}} \mathcal{I}_4^{1-\text{loop}}(s, t)$$

Agrees with known result of Green, Schwarz and Brink.

The two-particle cuts algebra recycles to all loop orders!
Loop Iteration of the Amplitude

Four-point one-loop $D = 4 - 2\epsilon$, $N = 4$ amplitude:

$$A^{1\text{-loop}}_4(s, t) = -stA^{\text{tree}}_4 I^{1\text{-loop}}(s, t)$$

$$I^{1\text{-loop}}(s, t) \sim \frac{1}{st} \left[ \frac{2}{\epsilon^2} \left( (-s)^{-\epsilon} + (-t)^{-\epsilon} \right) - \ln^2 \left( \frac{t}{s} \right) - \pi^2 \right] + \mathcal{O}(\epsilon)$$

To check for iteration use evaluation of two-loop integrals.

$$A^{2\text{-loop}}_4(1^-, 2^-, 3^+, 4^+) = -st A^{\text{tree}}_4 (1^-, 2^-, 3^+, 4^+) \left( s I^{2\text{-loop}}_4(s, t) + t I^{2\text{-loop}}_4(t, s) \right)$$

Integrals known and involve 4th order polylogarithms.

Planar contributions

Obtained via unitarity method

Bern, Rozowsky, Yan

V. Smirnov
Loop Iteration of the Amplitude

The planar four-point two-loop amplitude undergoes fantastic simplification.

Anastasiou, Bern, Dixon, Kosower

\[ M_{4}^{2\text{-loop}}(s,t) = \frac{1}{2} \left( M_{4}^{1\text{-loop}}(s,t) \right)^{2} + f(\epsilon) M_{4}^{1\text{-loop}}(s,t) \bigg|_{\epsilon \rightarrow 2\epsilon} - \frac{1}{2} \zeta_{2}^{2} \]

where

\[ M_{4}^{\text{loop}} = A_{4}^{\text{loop}}/A_{4}^{\text{tree}}, \quad f(\epsilon) = -\zeta_{2} - \zeta_{3} \epsilon - \zeta_{4} \epsilon^{2} \]

\( f(\epsilon) \) is universal function related to IR singularities

Thus we have succeeded to express two-loop four–point planar amplitude as iteration of one-loop amplitude.

Recent confirmation directly on integrands.

Cachazo, Spradlin and Volovich

\( D = 4 - 2\epsilon \)
Generalization to \( n \) Points

Can we guess the \( n \)-point result? Expect simple structure.

**Trick:** use collinear behavior for guess

Have calculated two-loop splitting amplitudes.

Following ansatz satisfies all collinear constraints

\[
M_{n}^{2\text{-loop}}(\epsilon) = \frac{1}{2} \left( M_{n}^{1\text{-loop}}(\epsilon) \right)^{2} + f(\epsilon) M_{n}^{1\text{-loop}}(2\epsilon) - \frac{1}{2} \zeta_{2}^{2}
\]

\[
M_{n}^{\text{loop}} = \frac{A_{n}^{\text{loop}}}{A_{n}^{\text{tree}}}, \quad f(\epsilon) = -\zeta_{2} - \zeta_{3} \epsilon - \zeta_{4} \epsilon^{2}
\]

Valid for planar MHV amplitudes

\[ D = 4 - 2\epsilon \]

Has correct analytic properties
As a non-trivial consistency check, worked out 5-point two-loop amplitudes. Cachazo, Spradlin and Volovich Bern, Czakon, Kosower, Roiban, Smirnov

To deal with the loop integrals we used Czakon’s wonderful MB integration package.

High precision numerical confirmation of iteration. Analytic proof would be better.
Three-loop Generalization

From unitarity method we get three-loop planar integrand:

\[-i s t A_{4}^{\text{tree}} \left\{ \frac{s^{2}}{3} + s(\ell + k_{2})^{2} + s(\ell + k_{4})^{2} \right\} \]

Use Mellin-Barnes integration technology and apply hundreds of harmonic polylog identities:

\[M_{4}^{3-\text{loop}}(\epsilon) = -\frac{1}{3} \left[ M_{4}^{1-\text{loop}}(\epsilon) \right]^{3} + M_{4}^{1-\text{loop}}(\epsilon) M_{4}^{2-\text{loop}}(\epsilon) + f^{3-\text{loop}}(\epsilon) M_{4}^{1-\text{loop}}(3 \epsilon) + C^{(3)} + \mathcal{O}(\epsilon)\]

where

\[f^{3-\text{loop}}(\epsilon) = \frac{11}{2} \zeta_{4} + \epsilon(6\zeta_{5} + 5\zeta_{2}\zeta_{3}) + \epsilon^{2}(c_{1}\zeta_{6} + c_{2}\zeta_{3}^{2}),\]

and

\[C^{(3)} = \left( \frac{341}{216} + \frac{2}{9}c_{1} \right) \zeta_{6} + \left( -\frac{17}{9} + \frac{2}{9}c_{2} \right) \zeta_{3}^{2}.\]

Answer actually does not actually depend on \(c_{1}\) and \(c_{2}\). Five-point calculation would determine these.
Three-loop Generalization to $n$ Points

Anastasiou, Bern, Dixon, Kosower

Repeat two-loop discussion, but at three loops.

Although we haven’t calculated the three-loop splitting function, by now it is clear it too should iterate.

Same logic as at two loops immediately gives three-loop generalization:

$$M_{n}^{3\text{-loop}}(\epsilon) = -\frac{1}{3} \left[ M_{n}^{1\text{-loop}}(\epsilon) \right]^{3} + M_{n}^{1\text{-loop}}(\epsilon) M_{n}^{2\text{-loop}}(\epsilon) + f^{3\text{-loop}}(\epsilon) M_{n}^{1\text{-loop}}(3 \epsilon) + C^{(3)} + O(\epsilon)$$

Valid for planar MHV amplitudes
Why not be bold and guess scattering amplitudes for all loop all legs (at least for MHV amplitudes)?

- Remarkable formula from Magnea and Sterman tells us IR singularities to all loop orders. Guides construction.
- Collinear limits gives us the key analytic information, at least for MHV amplitudes.

$$M_n = \exp \left[ \sum_{l=1}^{\infty} a^l \left( f^{(l)}(\epsilon) M_n^{(1)}(l\epsilon) + C^{(l)} + E_n^{(l)}(\epsilon) \right) \right]$$

$$a = \frac{N_C \alpha_s}{2\pi}$$

$$f^{(l)}(\epsilon) = f_0^{(l)} + \epsilon f_1^{(l)} + \epsilon^2 f_2^{(l)} + \cdots$$

- Soft anomalous dimension
- Or leading twist high spin anomalous dimension
- Or cusp anomalous dimension
- Or high moment limit of Altarelli-Parisi splitting kernel
After subtracting IR singularities finite remainder of the all loop order planar amplitude is:

\[ F_n = \exp \left[ \frac{1}{4} \gamma_K F^{(1)}_n + C \right] \]

All loop resummation of finite remainder

An unnamed constant

One-loop finite remainder. Complicated function of kinematic variables

Soft or cusp anomalous dimension

Same anomalous dimension guessed by Eden and Staudacher using integrability

\[ \gamma_K = \sum_{l=1}^{\infty} a^{(l)} \gamma^{(l)}_K \]

It seems likely that the simplicity uncovered here is connected to integrability.
Finite Remainder

\[ F_n = \exp \left[ \frac{1}{4} \gamma_K F_n^{(1)} + C \right] \]

\[ F_n^{(1)}(0) = \frac{1}{2} \sum_{i=1}^{n} g_{n,i} \]

\[ g_{n,i} = - \sum_{r=2}^{[n/2]-1} \ln \left( \frac{-t_i^{[r]}}{-t_i^{[r+1]}} \right) \ln \left( \frac{-t_{i+1}^{[r]}}{-t_i^{[r+1]}} \right) + D_{n,i} + L_{n,i} + \frac{3}{2} \zeta_2 \]

\[ D_{2m+1,i} = - \sum_{r=2}^{m-1} \text{Li}_2 \left( 1 - \frac{t_i^{[r]} t_i^{[r+1]}}{t_i^{[r+1]} t_i^{[r+1]}} \right) \]

\[ L_{2m+1,i} = - \frac{1}{2} \ln \left( \frac{-t_i^{[m]}}{-t_i^{[m]} - t_i^{[m+1]}} \right) \ln \left( \frac{-t_i^{[m]}}{-t_i^{[m]} - t_i^{[m+1]}} \right) \]

- All loop resummation of a one-loop amplitude in planar limit.
- In QCD this type of function contributes to physical quantities such as jet rates.
- IR divergences cancel against similar divergences from real emission diagrams.
It is suspected that $N = 4$ super-Yang-Mills is integrable in the planar limit. Recent proposal for soft/cusp anomalous dimension in $N = 4$ SYM to all perturbative orders, based on integrability.

Minhan and Zarembo; Beisert, Krisjansen, Staudacher and many others

Eden, Staudacher, hep-ph/0603157

Generating function for $\gamma_K$:

$$f(g) = 4g^2 - \frac{2}{3}\pi^2 g^4 + \frac{11}{45}\pi^4 g^6 - \left(\frac{73}{630}\pi^6 - 4\zeta(3)^2\right)g^8$$

$$+ \left(\frac{887}{14175}\pi^8 - \frac{4}{3}\pi^2\zeta(3)^2 - 40\zeta(3)\zeta(5)\right)g^{10}$$

$$- \left(\frac{136883}{3742200}\pi^{10} - \frac{8}{15}\pi^4\zeta(3)^2 - \frac{40}{3}\pi^2\zeta(3)\zeta(5)
- 210\zeta(3)\zeta(7) - 102\zeta(5)^2\right)g^{12}$$

$$+ \ldots.$$

Satisfies maximal transcendentality conjecture to be discussed by Lipatov

If we know soft anomalous dimension then we know finite remainder of all-loop MHV planar amplitudes, up to overall constant.
Is ES Conjecture Correct?

On path of checking our iteration formula at four loops we will extract the 4-loop anomalous dimension. $\gamma_K$ appears in coefficient of $1/\epsilon^2$ IR singularity.

We are in the midst of computing this: stay tuned…
At tree level Kawai, Lewellen and Tye have presented a relationship between closed and open string amplitudes. In field theory limit relationship is between gravity and gauge theory where we have stripped all coupling constants

\[ M_{4}^{\text{tree}}(1, 2, 3, 4) = s_{12} A_{4}^{\text{tree}}(1, 2, 3, 4) A_{4}^{\text{tree}}(1, 2, 4, 3), \]

\[ M_{5}^{\text{tree}}(1, 2, 3, 4, 5) = s_{12} s_{34} A_{5}^{\text{tree}}(1, 2, 3, 4, 5) A_{5}^{\text{tree}}(2, 1, 4, 3, 5) \]

\[ + s_{13} s_{24} A_{5}^{\text{tree}}(1, 3, 2, 4, 5) A_{5}^{\text{tree}}(3, 1, 4, 2, 5) \]

where we have stripped all coupling constants

\[ A_{4}^{\text{tree}} = g^2 \sum_{\text{non-cyclic}} \text{Tr}(T^{a_1}T^{a_2}T^{a_3}T^{a_4}) A_{4}^{\text{tree}}(1, 2, 3, 4) \]

Holds for any external states. See review: gr-qc/0206071

Progress in gauge theory can be imported into gravity theories.
Divergences in Supergravity

Conventional wisdom states that it impossible to construct a finite quantum field theory of gravity

- Flaw with *all* previous studies of divergences. Rely on powercounting, taking into account only supersymmetry.
- We now have a much deeper understanding: hidden structures, dualities, twistors, connection to sYM via KLT.
- Perturbative $N = 8$ supergravity inherits its property from $N = 4$ sYM.

*Is it finite, contrary to prevailing wisdom?*

Suppose we wanted to check this with Feynman diagrams:

First potential divergence is at 5 loops
This single diagram has $\sim 10^{30}$ terms prior to evaluating any integrals.
Impossible to evaluate via diagrams!
We may use KLT relations in conjunction with the unitarity method to check the divergence structure of gravity theories. Strategy already used to demonstrate that $N = 8$ sugra is less divergent than previously thought. First potential divergence will be at least 5 loops!

Bern, Dixon, Dunbar, Rozowsky, and Yan; Howe and Stelle

Similar twistor structures exist in gravity as in gauge theory.

Witten; Bern, Bjerrum-Bohr, Dunbar
Summary

- Motivation for studying amplitudes:
  (a) LHC demands QCD loop calculations.
  (b) Can we solve (planar) $N = 4$ super-Yang-Mills?
  (c) Is $N = 8$ supergravity finite, contrary to accepted wisdom? Demands explicit computations.
- Inspiration from twistor space: amazing simplicity.
- On-shell methods – unitarity and factorization.
- Explicit conjecture for resumming the MHV amplitudes of $N = 4$ super-Yang-Mills theory to all loop orders.
- Expect to have $N = 4$ four-loop soft anomalous dimension soon.

There are a variety of exciting avenues for further exploration of amplitudes in QCD, super-Yang-Mills theory and supergravity.
Extra Transparancies
Tree-level example: Five gluons

Consider the five-gluon amplitude

If you evaluate follow the textbooks you find…
Result of evaluation (actually only a small part of it):

\[ k_1 \cdot k_4 \epsilon_2 \cdot k_1 \epsilon_1 \cdot \epsilon_3 \epsilon_4 \cdot \epsilon_5 \]
A Remarkable Twistor String Formula

The following formula encapsulates the entire tree-level S-matrix of $N = 4$ super-Yang-Mills:

$$A_n = i(2\pi)^4 g_{\text{YM}}^{n-2} \sum_{d=1}^{n-3} \int \mathcal{M}_{n,d} \prod_{i=1}^{n} \delta^2 (\lambda_i^\alpha - \xi_i P_i^\alpha) \prod_{k=0}^{d} \delta^2 \left( \sum_{i=1}^{n} \xi_i \sigma_i^k \widetilde{\chi}_i^\alpha \right) \delta^4 \left( \sum_{i=1}^{n} \xi_i \sigma_i^k \eta_{iA} \right)$$

$$P_i^\alpha = \sum_{k=0}^{d} a_k^\alpha \sigma_i^k$$

Integral over the Moduli and curves

Strange formula from Feynman diagram viewpoint.

But it’s true: impressive checks by Roiban, Spradlin and Volovich
Early On-Shell Bootstrap

Bern, Dixon, Kosower (1997)

Early Approach:

- Use factorization properties to find rational function contributions.

Key problems preventing widespread applications:
- Difficult to find rational functions with desired factorization properties.
- Systematization unclear – key problem.
Other theories

Two classes of (large $N_c$) conformal gauge theories “inherit” the same large $N_c$ perturbative amplitude properties from $N=4$ SYM:

1. Theories obtained by orbifold projection – product groups, matter in particular bi-fundamental rep’s

   Bershadsky, Johansen, hep-th/9803249

2. The $N=1$ supersymmetric “beta-deformed” conformal theory – same field content as $N=4$ SYM, but superpotential is modified:

   \[ i g \text{Tr}(\Phi_1 \Phi_2 \Phi_3 - \Phi_1 \Phi_3 \Phi_2) \rightarrow \text{Tr}(e^{i \pi \beta R} \Phi_1 \Phi_2 \Phi_3 - e^{-i \pi \beta R} \Phi_1 \Phi_3 \Phi_2) \]

   Leigh, Strassler, hep-th/9503121

   Supergravity dual known for this case, deformation of $\text{AdS}_5 \times S^5$

   Lunin, Maldacena, hep-th/0502086

Breakdown of inheritance at five loops (!?) for more general marginal perturbations of $N=4$ SYM?

Khoze, hep-th/0512194
Beyond three loops

Recent proposal for soft/cusp anomalous dimension in N=4 SYM to all perturbative orders (!), based on integrability. Eden, Staudacher, hep-ph/0603157

\[ f(g) = 4g^2 - 16g^4 \int_0^\infty dt \hat{\sigma}(t) \frac{J_1(\sqrt{2}gt)}{\sqrt{2}gt} \]

where

\[ \hat{\sigma}(t) = \frac{t}{e^t - 1} \left[ \frac{J_1(\sqrt{2}gt)}{\sqrt{2}gt} - 2g^2 \int_0^\infty dt' \hat{K}(\sqrt{2}gt, \sqrt{2}gt') \hat{\sigma}(t') \right] \]

is the solution to an integral equation with Bessel-function kernel

\[ \hat{K}(t, t') = \frac{J_1(t) J_0(t') - J_0(t) J_1(t')}{t - t'} \]

Perturbative expansion:

\[ f(g) = 4g^2 - \frac{2}{3} \pi^2 g^4 + \frac{11}{45} \pi^4 g^6 - \left( \frac{73}{630} \pi^6 - 4 \zeta(3)^2 \right) g^8 + \ldots \]
Progress Towards the Dream

Results with on-shell methods:

• Complete QCD amplitudes with $n > 5$ legs.  
  see David Darren Forde’s talk
• Logarithmic contributions via on-shell recursion.
• Improved ways to obtain logarithmic contributions 
  via unitarity method. All six-gluon helicities.

Key Feature: Modest growth in complexity as $n$ increases.
Loop-Level Recursion

New Features:

- Presence of branch cuts.
- Unreal poles – poles appear with complex momenta.
  \[
  \frac{[a \ b]}{\langle a \ b \rangle}
  \]
  Pure phase for real momenta

- Double poles.
  \[
  \frac{[a \ b]}{\langle a \ b \rangle^2}
  \]

- Spurious singularities that cancel only against polylogs.

- Double count between cuts and recursion.

See Carola Berger’s and Darren Forde’s talks
The Gold Standard: NNLO Drell-Yan Rapidity Distributions

- Amazingly good stability
- Theoretical uncertainties less than 1%
• One of the most remarkable discoveries in elementary particle physics has been that of the existence of the complex plane

J. Schwinger in “Particles, Sources and Fields” Vol 1