The S-Matrix of AdS/CFT

Niklas Beisert

Princeton University

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* in collaboration with M. Staudacher
Introduction

AdS/CFT Conjecture

- $\mathcal{N} = 4$ gauge theory (exactly) dual to IIB superstrings on $AdS_5 \times S^5$.
- Spectra should agree. Would like to test.
- Weak coupling: Planar gauge theory states described by a spin chain.
- Strong coupling: Planar quantum strings described by sigma model.
- Both models appear integrable. Make full use.

Outline

- 1D Particle model from planar string and gauge theory.
- Nature of the Hamiltonian.
- Symmetry: Centrally extended $su(2|2)$.
- Construction of S-matrix, Yang-Baxter equation.
- Relation to Shastry’s R-matrix for the Hubbard chain.
- S-matrix of AdS/CFT. Abelian phase.
Planar IIB Superstrings on $\text{AdS}_5 \times S^5$

IIB superstrings propagate on the curved superspace $\text{AdS}_5 \times S^5$

Subspaces

$$S^5 = \frac{\text{SO}(6)}{\text{SO}(5)} = \frac{\text{SU}(4)}{\text{Sp}(2)}, \quad \text{AdS}_5 = \frac{\widetilde{\text{SO}}(2, 4)}{\text{SO}(1, 4)} = \frac{\widetilde{\text{SU}}(2, 2)}{\text{Sp}(1, 1)}, \quad \text{fermi} = \mathbb{R}^{32}.$$ 

Coset space

$$\text{AdS}_5 \times S^5 \times \text{fermi} = \frac{\widetilde{\text{PSU}}(2, 2|4)}{\text{Sp}(1, 1) \times \text{Sp}(2)}.$$
The $\mathcal{N} = 4$ Spin Chain and $\mathfrak{psu}(2, 2|4)$

Basis of spins: All fields of the theory & derivatives

$\mathcal{W} \in \{D^n\Phi, D^n\Psi, D^nF\}$.

One irreducible non-compact lowest-weight module $\mathcal{V}_F$ of $\mathfrak{psu}(2, 2|4)$.

Basis of states: Single-trace operators (cyclicity of gauge invariant states)

$\mathcal{O} = \text{Tr} \mathcal{W}_1 \mathcal{W}_2 \mathcal{W}_3 \ldots \mathcal{W}_L = \pm \text{Tr} \mathcal{W}_2 \mathcal{W}_3 \ldots \mathcal{W}_L \mathcal{W}_1 = \ldots$.

Spin chain with non-compact spin representation $\mathcal{V}_F$
Hamiltonian as a Symmetry Generator

The Hamiltonian . . .

• . . . generates time translations on $AdS_5$,
• . . . is an isometry of the string background,
• . . . is part of superconformal symmetry,
• . . . is part of the symmetry algebra $\tilde{\text{PSU}}(2, 2|4)$.

Unusual spin chain model where . . .

• . . . the Hamiltonian does not commute with the symmetry generators,
• . . . the Hamiltonian is part of the non-abelian symmetry algebra,
• . . . energies determine how states transform under symmetry,
• . . . energies are labels for multiplets of states,
• . . . the multiplet structure is dynamical,
• . . . the multiplets have a continuous (energy) label.

Split $\mathcal{H}(g) = \mathcal{H}_0 + \delta\mathcal{H}(g)$: Classical energy $\mathcal{H}_0$ and energy shift $\delta\mathcal{H}(g)$. 

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Light Cone Gauge and a Particle Model

Perform light cone gauge using time from $AdS_5$ and great circle from $S^5$.

- **Vacuum**: Point-particle moving along time and great circle.
- **Excitations**: 4 coordinates on $AdS_5$ and 4 coordinates on $S^5$.
- **Fermions**: 1/2 are momenta, 1/2 are gauged away, 8 remain.

QM particle model of 8 bosonic and 8 fermionic flavours on the circle.

Residual symmetry

Unbroken symmetries of the vacuum state:

- $\widetilde{SO}(4, 2) \simeq \widetilde{SU}(2, 2)$ reduces to $SO(4) \simeq SU(2) \times SU(2)$.
- $SO(6) \simeq SU(4)$ reduces to $SO(4) \simeq SU(2) \times SU(2)$.
- $PSU(2, 2|4)$ reduces to $(PSU(2|2) \times PSU(2|2)) \rtimes \mathbb{R}$.

Excitations transform in representations of $(PSU(2|2) \times PSU(2|2)) \rtimes \mathbb{R}$. 
Coordinate Space Bethe Ansatz

Consider spin chain states with few “excitations”.

Ferromagnetic vacuum of an infinite chain: protected state with scalar $Z$

$$|0\rangle = |...Z Z Z...\rangle, \quad \delta \mathcal{H} |0\rangle = 0.$$ 

One-excitation states with excitation $A$ at position $a$, momentum $p$

$$|A(p)\rangle = \sum_a e^{ipa}|...\dot{\phi}...\dot{\psi}...\rangle, \quad \delta \mathcal{H} |A(p)\rangle = \delta E_A(p)|A(p)\rangle.$$ 

(4+4|+4) flavours of single excitations $A \in \{\phi^i, D^\mu Z|\dot{\psi}^a, \dot{\dot{\psi}}^\dot{a}\}$. [Berenstein Maldacena Nastase]

The remaining (infinitely many) spin orientations in module $V_F$

• are multiple excitations,

• arise when two or more elementary excitations coincide on a single site.

Coordinate space Bethe ansatz leads to a particle model with 8|8 flavours.
Residual Symmetry

1D particle model with particles of $8|8$ flavours.

States transform under $(30|32)$ generators of $\mathfrak{psu}(2,2|4)$.

- $(8|8)$ generators create excitations with $p = 0$,
- $(8|8)$ generators annihilate excitations with $p = 0$,
- $(14|16)$ remaining generators transform the excitations.

Regroup excitations

$$\mathcal{A} \in \left\{ \begin{array}{ccc|ccc} \phi^{11} & \phi^{12} & \psi^{11} & \psi^{12} \\
\phi^{21} & \phi^{22} & \psi^{21} & \psi^{22} \\
\psi^{11} & \psi^{12} & \mathcal{D}^{11} & \mathcal{D}^{12} \\
\psi^{21} & \psi^{22} & \mathcal{D}^{21} & \mathcal{D}^{22} \end{array} \right\}.$$

Residual symmetry

$$\mathbb{R} \ltimes (\mathfrak{psu}(2|2) \times \mathfrak{psu}(2|2)) \ltimes \mathbb{R}.$$

Hamiltonian $\mathcal{H}$ is central charge of both $\mathfrak{psu}(2|2) \ltimes \mathbb{R}$.
**$\mathfrak{su}(2|2)$ Residual Algebra**

1D particle model with $2|2$ flavours $(\phi^a, \psi^\alpha)$. Inhomogeneous spin chain.

Residual algebra $\mathfrak{su}(2|2)$ preserves particle number. Generators:

- $\mathcal{R}^a_b$ internal $\mathfrak{su}(2)$ rotation generator,
- $\mathcal{L}^\alpha_\beta$ spacetime $\mathfrak{su}(2)$ rotation generator,
- $\mathcal{Q}^\alpha_b$ supersymmetry generator,
- $\mathcal{S}^a_\beta$ superboost generator,
- $\mathcal{C}$ central charge. Particle number and energy: $C = \frac{1}{2} K + \frac{1}{2} \delta E$.

Algebra: $\mathcal{R}^a_b, \mathcal{L}^\alpha_\beta$ transform indices. Anticommutator of supercharges

\[
\{\mathcal{Q}_a^\alpha, \mathcal{S}_b^\beta\} = \delta^b_a \mathcal{L}_\beta^\alpha + \delta^\alpha_\beta \mathcal{R}_a^b + \delta^b_a \delta^\alpha_\beta \mathcal{C},
\]

\[
\{\mathcal{Q}_a^\alpha, \mathcal{Q}_b^\beta\} = 0,
\]

\[
\{\mathcal{S}_a^\alpha, \mathcal{S}_b^\beta\} = 0.
\]

Excitations should transform in $(2|2)$ representation of $\mathfrak{su}(2|2)$. 
Extension to \( \mathfrak{su}(2|2) \ltimes \mathbb{R}^2 \)

Problems:

- Central charge acts as energy: \( C = \frac{1}{2}K + \frac{1}{2}\delta E \) with \( \delta E \) continuous.
- The only \((2|2)\)-dimensional representations are the fundamentals.
- The fundamental representations have \( C = \pm\frac{1}{2} \).
- The asymptotic states have discrete \( C \): No anomalous dimensions?

Resolution:

- Enlarge algebra by two central charges \( \mathfrak{P}, \mathfrak{K} \): \( \mathfrak{su}(2|2) \ltimes \mathbb{R}^2 \) [hep-th/0511082]

\[
\{ \mathfrak{Q}^\alpha_a, \mathfrak{Q}^\beta_b \} = \varepsilon^{\alpha\beta} \varepsilon_{ab} \mathfrak{P}, \quad \{ \mathfrak{S}^a_\alpha, \mathfrak{S}^b_\beta \} = \varepsilon^{ab} \varepsilon_{\alpha\beta} \mathfrak{K}.
\]

- Closure of algebra for the \((2|2)\) representation requires

\[
\vec{C}^2 = C^2 - PK = \frac{1}{4}.
\]

- Family of \((2|2)\) representations with continuous \( C \) (mass shell).
Fundamental Representation of $su(2|2) \rtimes \mathbb{R}^2$

Ansatz for $(2|2)$ representation with canonical action of $\mathcal{R}_a^b, \mathcal{L}^\alpha{}_\beta$:

\begin{align*}
\mathcal{Q}^\alpha{}_a |\phi^b\rangle &= a \delta^b_a |\psi^\alpha\rangle, \\
\mathcal{Q}^\alpha{}_a |\psi^\beta\rangle &= b \varepsilon^{\alpha\beta} \varepsilon_{ab} |\phi^b Z^+\rangle, \\
\mathcal{S}^a{}_\alpha |\phi^b\rangle &= c \varepsilon^{ab} \varepsilon_{\alpha\beta} |\psi^\beta Z^-\rangle, \\
\mathcal{S}^a{}_\alpha |\psi^\beta\rangle &= d \delta^\beta_\alpha |\phi^a\rangle, \\
\mathcal{C}|X\rangle &= C |X\rangle, \\
\mathcal{P}|X\rangle &= P |X Z^+\rangle, \\
\mathcal{K}|X\rangle &= K |X Z^-\rangle.
\end{align*}

Closure requires $ad - bc = 1, \ C = \frac{1}{2}(ad + bc), \ P = ab, \ K = cd$.

Denote this multiplet by $\langle \vec{C} \rangle = \langle C, P, K \rangle$. 
Reduction to $\mathfrak{su}(2|2)$ for Composite States

To recover $\mathfrak{su}(2|2)$ we need $\mathcal{P}$, $\mathcal{K}$ to annihilate physical states.

Idea: Identify them with gauge transformations $\delta \mathcal{W} = [\epsilon, \mathcal{W}]$.

For $\mathcal{P}$ use gauge parameter $\epsilon = g\alpha \mathcal{Z}$.

• Vacuum spin $\mathcal{Z}$ trivially invariant: $\mathcal{P}|0\rangle = 0$.
• Particle transforms as $\mathcal{P}|\mathcal{X}\rangle = g\alpha |[\mathcal{Z}^+, \mathcal{X}]\rangle = g\alpha |\mathcal{Z}^+ \mathcal{X}\rangle - g\alpha |\mathcal{X} \mathcal{Z}^+\rangle$.

Symbols $\mathcal{Z}^\pm$ mean insertion/removal of vacuum site $\mathcal{Z}$. Implies

$$|\mathcal{Z}^\pm \mathcal{X}\rangle = e^{\mp ip} |\mathcal{X} \mathcal{Z}^\pm\rangle.$$

Likewise, $\mathcal{K}|\mathcal{X}\rangle = g\alpha^{-1} |[\mathcal{Z}^-, \mathcal{X}]\rangle$ to remove vacuum site. Action of $\mathcal{P}$, $\mathcal{K}$:

$$\mathcal{P}|\mathcal{X}\rangle = P |\mathcal{Z}^+ \mathcal{X}\rangle, \quad \mathcal{K}|\mathcal{K}\rangle = K |\mathcal{Z}^- \mathcal{X}\rangle$$

with

$$P = g\alpha (1 - e^{+ip}), \quad K = g\alpha^{-1} (1 - e^{-ip}).$$

Cyclicity condition $P = K = 0$ for physical states with zero momentum.
Solution for the Representation

Momentum constraint and energy eigenvalue $C$ (central charge)

\[ \prod_{k=1}^{K} e^{ip_k} = 1, \quad C = \sum_{k=1}^{K} C_k, \quad C_k = \pm \frac{1}{2} \sqrt{1 + 16g^2 \sin^2\left(\frac{1}{2}p_k\right)}. \]

Introduce spectral parameters $x_k^\pm$ with

\[ x_k^+ + \frac{1}{x_k^+} - x_k^- - \frac{1}{x_k^-} = \frac{i}{g}. \]

Then momentum and energy of an excitation read

\[ e^{ip_k} = \frac{x_k^+}{x_k^-}, \quad C_k = \frac{1}{2} + \frac{ig}{x_k^+} - \frac{ig}{x_k^-} = -igx_k^+ + igx_k^- - \frac{1}{2}. \]
S-Matrix as an Invariant Operator

S-matrix is a two-particle permutation operator

\[ S_{kl}|\ldots \chi_k \chi'_l \ldots \rangle \mapsto *|\ldots \chi''_l \chi'''_k \ldots \rangle. \]

invariant under \( su(2|2) \times \mathbb{R}^2 \): \([ \mathcal{J}_k + \mathcal{J}_l, S_{kl} ] = 0 \).

From manifest \( su(2) \times su(2) \) symmetries

\[
\begin{align*}
S_{12}|\phi^a_1 \phi^b_2 \rangle &= A_{12}|\phi^\{a\}_2 \phi^b_1 \rangle + B_{12}|\phi^\{a\}_2 \phi^b_1 \rangle + \frac{1}{2}C_{12}\varepsilon^{ab}\varepsilon_{\alpha\beta}|\psi^\alpha_2 \psi^\beta_1 \mathcal{Z}^-\rangle, \\
S_{12}|\psi^\alpha_1 \psi^\beta_2 \rangle &= D_{12}|\psi^\{\alpha\}_2 \psi^\beta_1 \rangle + E_{12}|\psi^\{\alpha\}_2 \psi^\beta_1 \rangle + \frac{1}{2}F_{12}\varepsilon^{\alpha\beta}\varepsilon_{ab}|\phi^a_2 \phi^b_1 \mathcal{Z}^+\rangle, \\
S_{12}|\phi^a_1 \psi^\beta_2 \rangle &= G_{12}|\psi^\beta_2 \phi^a_1 \rangle + H_{12}|\phi^a_2 \psi^\beta_1 \rangle, \\
S_{12}|\psi^\alpha_1 \phi^b_2 \rangle &= K_{12}|\psi^\alpha_2 \phi^b_1 \rangle + L_{12}|\phi^b_2 \psi^\alpha_1 \rangle.
\end{align*}
\]

with ten coefficient functions \( A_{12}, \ldots, L_{12} \) and \( A_{12} = A(p_1, p_2), \ldots \)
Coefficients
Commutators with fermionic generators relate all coefficients:

\[A_{12} = S_{12}^0 \frac{x_2^+ - x_1^-}{x_2^- - x_1^+},\]
\[G_{12} = S_{12}^0 \frac{1}{\xi_1} \frac{x_2^+ - x_1^+}{x_2^- - x_1^+},\]
\[H_{12} = S_{12}^0 \frac{\xi_2 \gamma_1}{\xi_1} \frac{x_2^+ - x_2^-}{x_2^- - x_1^+},\]
\[K_{12} = S_{12}^0 \frac{\gamma_2}{\gamma_1} \frac{x_1^+ - x_1^-}{x_2^- - x_1^+},\]
\[L_{12} = S_{12}^0 \xi_2 \frac{x_2^- - x_1^-}{x_2^- - x_1^+},\]
\[D_{12} = -S_{12}^0 \frac{\xi_2}{\xi_1}.\]

Spectral parameters \(x_k^\pm = x^\pm(p_k)\) functions of particle momenta \(p_k\).

- Overall factor \(S_{12}^0 = S^0(p_1, p_2)\) cannot be determined by symmetry.
- S-matrix not of difference form \(S(p_1, p_2) \neq S(f(p_1) - f(p_2)).\)
Uniqueness and Representations

Why is the S-matrix uniquely determined by symmetry alone?

Consider the tensor product of two fundamentals

\[ \langle \vec{C} \rangle \otimes \langle \vec{C}' \rangle = \{0, 0; \vec{C} + \vec{C}'\}. \]

The one resulting representations is irreducible!

- Fundamental representation \( \langle \vec{C} \rangle \) is short/atypical: 4 components.
  \( \{0, 0; \vec{C}\} \) is long/typical (unless \( \vec{C}^2 = 1 \)): 16 components. \( 4 \times 4 = 16 \).
- Shortening condition for \( \{m, n; \vec{C}\} \) is quadratic \( \vec{C}^2 = \frac{1}{4}(n + m + 2)^2 \).
  Generally not met because \( (\vec{C} + \vec{C}')^2 = \frac{1}{2} + 2\vec{C} \cdot \vec{C}' \neq 1 \).
- Very unusual to obtain only one irrep in tensor product.
Unitarity

Does the S-matrix satisfy the unitarity condition $S_{21}S_{12} = 1$?

It maps between identical spaces

$$S_{21}S_{12} : \langle \vec{C}_1 \rangle \otimes \langle \vec{C}_2 \rangle \rightarrow \langle \vec{C}_1 \rangle \otimes \langle \vec{C}_2 \rangle.$$ 

But $\langle \vec{C}_1 \rangle \otimes \langle \vec{C}_2 \rangle = \{0, 0; \vec{C}_1 + \vec{C}_2\}$ is irreducible, so $S_{21}S_{12} \sim \mathcal{I}$!

Show $S_{21}S_{12} = \mathcal{I}$ for one component $|\psi^1\psi^1\rangle$: $S_{12}$ acts as factor $D_{12}$.

Overall factor: $D_{12}D_{21} = S_{12}^0S_{21}^0$. Unitarity requires $S_{12}^0S_{21}^0 \overset{!}{=} 1$. 
Yang-Baxter Relation

Does it satisfy the Yang-Baxter equation $S_{12}S_{13}S_{23} = S_{23}S_{13}S_{12}$?

Consider the tensor product

$$\langle \vec{C}_1 \rangle \otimes \langle \vec{C}_2 \rangle \otimes \langle \vec{C}_3 \rangle = \{0, 1; \vec{C}_1 + \vec{C}_2 + \vec{C}_3\} \oplus \{1, 0; \vec{C}_1 + \vec{C}_2 + \vec{C}_3\}.$$

Then show $S_{21}S_{31}S_{32}S_{12}S_{13}S_{23} = \mathcal{I}$ for one representative state in each multiplet to prove YBE: $|\phi^1\phi^1\phi^1\rangle$ and $|\psi^1\psi^1\psi^1\rangle$. No mixing here! S-matrix $S_{12}$ acts as factor $A_{12}$ and $D_{12}$, respectively. Trivially satisfied.
Nested Bethe Ansatz

To describe a state we started with

- a **homogeneous spin chain** infinitely many spin orientations

\[ |...ZZ\phi^aZZZZ\phi^bZZZZ...ZZ\psi^cZZZ...\rangle^0. \]

We can now describe a state by

- a set of \( K \) momenta \( p_k \),
- an **inhomogeneous spin chain** with \((2|2)\) spin orientations

\[ |\phi_1^a\phi_2^b\psi_3^c\rangle^1. \]

We should repeat this procedure to trade more orientations for momenta.

In the end, we would like to describe a state by **continuous numbers only**

- a set of \( K \) parameters \( p_k \) (expressed through \( x_k^{\pm} \)),
- a set of \( N \) parameters \( y_k \),
- a set of \( M \) parameters \( w_k \).

Achieved by **nested Bethe ansatz**.
Diagonalised Scattering

Components of the diagonalised S-matrix

\[ S_{12}^{I,0} = \frac{x_1^+}{x_1^-}, \quad S_{12}^{II,1} = \xi_2 \frac{y_1 - x_2^-}{y_1 - x_2^+}, \quad S_{12}^{III,II} = \frac{w_1 - y_2 - 1/y_2 - ig^{-1}}{w_1 - y_2 - 1/y_2 + ig^{-1}}, \]

\[ S_{12}^{I,1} = S_{12}^0 \frac{x_1^- - x_2^+}{x_1^+ - x_2^-}, \quad S_{12}^{II,II} = 1, \quad S_{12}^{III,III} = \frac{w_1 - w_2 + ig^{-1}}{w_1 - w_2 - ig^{-1}}. \]

Periodicity condition: Bethe equation

\[ 1 = \prod_{j=1}^{K+M+N} S_{jk}. \]
Hubbard Chain

- Spin chain model of 2 bosonic and 2 fermionic spin orientations.
- It has $\mathfrak{su}(2) \times \mathfrak{su}(2)$ symmetry (for odd length only $\mathfrak{su}(2)$).
- It has a coupling constant $U$.
- It does not fit into scheme of conventional integrable spin chains.
- Shastry's R-matrix for this model is not of difference form.

Is there a relationship? Compare to Lieb-Wu equations:

\begin{align*}
1 &= \prod_{j=1}^{K} \frac{y_k - x_j^+}{\xi_j} \prod_{j=1}^{M} \frac{y_k + 1/y_k - w_j + \frac{i}{2}g^{-1}}{y_k + 1/y_k - w_j - \frac{i}{2}g^{-1}}, \\
1 &= \prod_{j=1}^{N} \frac{w_k - y_j - 1/y_j + \frac{i}{2}g^{-1}}{w_k - y_j - 1/y_j - \frac{i}{2}g^{-1}} \prod_{j=1}^{M} \frac{w_k - w_j - ig^{-1}}{w_k - w_j + ig^{-1}}, \\
1 &= \exp(-ik_kK) \prod_{j=1}^{M} \frac{2\sin k_k - 2\Lambda_j + \frac{i}{2}U}{2\sin k_k - 2\Lambda_j - \frac{i}{2}U}, \\
1 &= \prod_{j=1}^{N} \frac{2\Lambda_k - \sin k_j + \frac{i}{2}U}{2\Lambda_k - \sin k_j - \frac{i}{2}U} \prod_{j=1}^{M} \frac{2\Lambda_k - 2\Lambda_j - iU}{2\Lambda_k - 2\Lambda_j + iU}.
\end{align*}

Need to identify

\[ g^{-1} = U, \quad w_k = 2\Lambda_k, \quad y_k = -i \exp(i k_k). \]

Furthermore, \( x_k^+ = i \xi_k, \ x_k^- = -i/\xi_k \) and take the limit $\xi_k \to \infty$.  

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Shastry’s R-matrix

Form of present S-matrix similar to Shastry’s R-matrix. Compare coeffs $A, \ldots, L$ to $\alpha_1, \ldots, 10$ from Ramos-Martins. Adjusting the parameters $x_k^\pm, \xi_k, \gamma_k$ to $a_k, b_k, h_k$ appropriately, we find

\[
\frac{\alpha_2}{\alpha_1} = \frac{A_{12}}{D_{12}}, \quad \frac{\alpha_3}{\alpha_1} = \frac{D_{12} + E_{12}}{2D_{12}}, \quad \frac{\alpha_4}{\alpha_1} = \frac{A_{12} + B_{12}}{2D_{12}},
\]
\[
\frac{\alpha_5}{\alpha_1} = \frac{H_{12}}{D_{12}} = \frac{K_{12}}{D_{12}}, \quad \frac{\alpha_6}{\alpha_1} = -\frac{D_{12} - E_{12}}{2D_{12}}, \quad \frac{\alpha_7}{\alpha_1} = \frac{A_{12} - B_{12}}{2D_{12}},
\]
\[
\frac{\alpha_8}{\alpha_1} = \frac{G_{12}}{D_{12}}, \quad \frac{\alpha_9}{\alpha_1} = -\frac{L_{12}}{D_{12}}, \quad \frac{\alpha_{10}}{\alpha_1} = -\frac{C_{12}}{2D_{12}} = -\frac{F_{12}}{2A_{12}}.
\]

R-matrix is equivalent to present S-matrix (up to twist)!

- R-Matrix has hidden $\mathfrak{su}(2|2) \ltimes \mathbb{R}^2$ supersymmetry.
- Supersymmetry broken in Hubbard chain by choice of $x_k^\pm, \xi_k$.
- Hubbard chain related to exceptional Lie superalgebra $\mathfrak{d}(2|1; \alpha)$.
- AdS/CFT scattering related to R-matrix of Hubbard chain.
- Different from earlier relation by Rej-Serban-Staudacher.
The S-Matrix for AdS/CFT

Generalise to full symmetry $\text{psu}(2, 2|4)$ of AdS/CFT. Particles transform as $(2|2) \otimes (2|2)'$ under $(\text{psu}(2|2) \times \text{psu}(2|2)') \rtimes \mathbb{R}^3$

$$\langle \tilde{C} \rangle \otimes \langle \tilde{C}' \rangle.$$

Scattering of multiplets factorises. Complete S-matrix is a product

$$S_{12}^{\text{AdS/CFT}} = S_{12} S'_{12}.$$

Diagonalised excitations: Five types $w'_k, y'_k, p_k, y_k, w_k$.

Momenta $p_k$ shared between $\text{psu}(2|2)$'s.
Abelian Phase

The phase \( S^0(p_k, p_j) \) is unconstrained. Perturbatively (?)

\[
(S^0_{kj})^2 = \frac{1 - 1/x_k^- x_j^+}{1 - 1/x_k^+ x_j^-} \frac{x_k^+ - x_j^-}{x_k^- - x_j^+} \exp(-2i\theta_{kj}),
\]

\[
\theta_{kj} = \sum_{r=2}^{\infty} \sum_{s=r+1}^{\infty} \beta_{rs}(g^2) (q_{r,k} q_{s,j} - q_{s,k} q_{r,j}),
\]

\[
q_{r,k} = \frac{1}{r - 1} \left( \frac{i}{(x_k^+)^{r-1}} - \frac{i}{(x_k^-)^{r-1}} \right).
\]

- Proposal for gauge theory: \( \beta_{rs} = 0 \) from Feynman diagrams. Confirmed at three loops. Exact?!
  Works in non-compact sectors.

- Proposal for string theory: \( \beta_{rs} = g\delta_{r+1,s} + O(g^0) \).
  Corrections for sigma model loops needed.
Conclusions

★ Planar AdS/CFT Correspondence
• Exciting spin chain model from $\mathcal{N} = 4$ gauge theory.
• Strings theory & coordinate space Bethe ansatz for gauge theory:
  Particle model with $(2|2) \times (2|2)$ particle flavours.
• Residual symmetry is two copies of $\mathfrak{su}(2|2)$ with central extensions.
• Unique S-matrix constructed. YBE automatically satisfied.

★ Hubbard Chain
• R-matrix supersymmetric. Scattering in AdS/CFT like Hubbard chain.

★ Open Questions
• Prove integrability for gauge and string theory.
• Find abelian phase $S_{12}^0$ consistent with crossing symmetry.
• Understand better & generalise $\mathfrak{su}(2|2) \ltimes \mathbb{R}^2$ chain.