Feynman Integrals for QCD

Andreas v. Manteuffel

Amplitudes 2015
Multi-loop Feynman integrals

\[ I = \int \frac{d^d k_1}{i\pi^{d/2}} \cdots \frac{d^d k_L}{i\pi^{d/2}} \frac{1}{D_1^{a_1} \cdots D_N^{a_N}} \quad a_i \in \mathbb{Z}, \quad D_1 = k_1^2 - m_1^2 \text{ etc.} \]

linear dependencies:
- integration-by-parts (IBP) identities [Tkachov, Chetyrkin '81]
- systematic reduction to master integrals [Laporta '00]
- think of it as linear vector space with some arbitrary basis

this talk: two solving methods
1. differential equations in kinematic invariants
2. direct integration of Feynman (Schwinger) parameters

CHOICE OF BASIS
1. choose master integrals suitable for respective integration method
2. optimize functional basis for solution
Part I: Two-loop amplitudes for diboson production

[Gehrmann, AvM, Tancredi, Weihs]
**Vector boson pair production at LHC**

\[ pp \rightarrow VV' + X \rightarrow 4 \text{ leptons} + X, \quad \text{where } VV' = ZZ, \ W^+ W^-, \ \gamma^* \gamma^*, \ Z W^\pm, \ Z \gamma^*, \ W^\pm \gamma^* \]

- sensitive to details of EWSB
- possible NP contributions at tree or loop level

e.g. \( W^+ W^- \) production:

![Feynman diagrams for \( W^+ W^- \) production](image)

**important background to Higgs signals:**

![Feynman diagrams for Higgs production](image)
2014 excess in $WW$ production at LHC

$\int L dt = 20.3 \text{ fb}^{-1}$

$\sqrt{s} = 8 \text{ TeV}$

$WW$

**Measured cross sections**

- $e^+e^-$
- $\mu^+\mu^-$
- $e^\pm\mu^\mp$
- Combined

**SM Prediction**

- $qq/gg \rightarrow WW$: MCFM NLO CT10
- $gg \rightarrow WW$: MCFM LO CT10
- $gg \rightarrow H \rightarrow WW$: NNLO MSTW2008

**Feynman integrals for QCD**

**Amplitudes 2015**

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*new physics?*

- make sure to understand SM prediction!
ingredients for $VV' + X$ production at NNLO QCD:

<table>
<thead>
<tr>
<th></th>
<th>LO</th>
<th>NLO</th>
<th>NNLO</th>
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<tr>
<td>$2 \to 2$</td>
<td>$\mathcal{M}_0^* \mathcal{M}_0$ $qq$</td>
<td>$\mathcal{M}_0^* \mathcal{M}_1$ $qq$</td>
<td>$\mathcal{M}_0^* \mathcal{M}_2, \mathcal{M}_1^* \mathcal{M}_1$ $qq, gg$</td>
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<tr>
<td>$2 \to 3$</td>
<td>- $\mathcal{M}_0^* \mathcal{M}_0$ $qq, qg$</td>
<td>$\mathcal{M}_0^* \mathcal{M}_1$ $qq, qg$</td>
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<tr>
<td>$2 \to 4$</td>
<td>-</td>
<td>-</td>
<td>$\mathcal{M}_0^* \mathcal{M}_0$ $qq, qg, gg$</td>
</tr>
</tbody>
</table>

Note: some channels contribute only at higher orders:
- $qg$ starting at NLO
- $gg$ starting at NNLO $\rightarrow$ control error by computing $N^3$LO contributions from this channel

Subtraction terms: up to 2 unresolved partons needed
- $q_T$ subtraction: [Catani, Grazzini '07; Catani, Cieri, de Florian, Ferrera, Grazzini '13]
- $N$-jettiness subtraction: [Boughezal, Foecke, Liu, Petriello '15; Boughezal, Foecke, Giele, Liu, Petriello '15; Gaunt, Stahlhofen, Tackmann, Walsh '15]
- Antenna subtraction: [Gehrmann-De Ridder, Gehrmann, Glover '05]
- Sector-improved subtraction: [Czakon '10]
QCD approx. NNLO and electroweak NLO:
- \( gg \) initiated (one-loop only): [Binoth et al. ('05,'08); Duhrssen et al. ('05); Amettler et al. ('85); van der Bij, Glover ('88); Adamson, de Florian, Signer ('00)]
- high energy \( WW \): [Chachamis, Czakon, Eiras ('08)]
- electroweak NLO: [Hollik, Meier (2004); Accomando, Denner, Meier ('05); Bierweiler, Kasprzik, Kühn, Uccirati ('12); Baglio, Ninh, Weber ('13); Billoni, Dittmaier, Jäger, Speckner ('13)]

QCD full NNLO (equal masses):
- master integrals: [Gehrmann, Tancredi, Weihs '13; Gehrmann, AvM, Tancredi, Weihs '14]
- amplitudes: [Gehrmann, AvM, Tancredi (unpublished)]
  \( ZZ \)@NNLO [Cascioli, Gehrmann, Grazzini, Kallweit, Maierhöfer, AvM, Pozzorini, Rathlev, Tancredi, Weihs'14]
  \( WW \)@NNLO [Gehrmann, Grazzini, Kallweit, Maierhöfer, AvM, Pozzorini, Rathlev, Tancredi '14]

QCD full NNLO (different masses):
- master integrals: [Henn, Melnikov, Smirnov '14; Caola, Henn, Melnikov, Smirnov '14]; [Papadopoulos, Tammasini, Wever '14]; [Gehrmann, AvM, Tancredi '15]
- amplitudes \( qq' \rightarrow VV' \): [Caola, Henn, Melnikov, Smirnov '14]; [Gehrmann, AvM, Tancredi '15]
- amplitudes \( gg \rightarrow VV' \): [Caola, Henn, Melnikov, Smirnov '15]; [AvM, Tancredi '15]
- \( \gamma^*\gamma^* \)@NNLO (partial): [Anastasiou, Cancino, Chavez, Duhr, Lazopoulos, Mistlberger, Müller '14]
- upcoming exact differential NNLO for various final states:
  see talks by [S. Kallweit, D. Rathlev, M. Wiesemann] at RadCor-LoopFest 2015
**Feynman diagrams** (generated with Qgraf [Nogueira])

$q\bar{q}'$ channel (just non-zero classes shown):

\[ [A] \quad [B] \]

\[ [C] \quad [F_V] \]

$gg$ channel ([B] and $[F_V]$ do not contribute):

\[ [A] \quad [B] \]

\[ [F_V] \]
\textbf{Lorentz structures for $VV'$ amplitude}

$VV'$ amplitude:

\[ S^{\mu\nu}(p_1, p_2, p_3) = \sum_j A_j(s, t, p_3^2, p_4^2) T_j^{\mu\nu} \]

$q\bar{q}'$ channel:

\[
T_1^{\mu\nu} = \bar{u}(p_2) \gamma_3 u(p_1) p_1^\mu p_2^{\nu}, \\
T_2^{\mu\nu} = \bar{u}(p_2) \gamma_3 u(p_1) p_1^\nu p_2^{\mu}, \\
T_3^{\mu\nu} = \bar{u}(p_2) \gamma_3 u(p_1) p_2^\mu p_1^{\nu}, \\
T_4^{\mu\nu} = \bar{u}(p_2) \gamma_3 u(p_1) p_2^\nu p_1^{\mu}, \\
T_5^{\mu\nu} = \bar{u}(p_2) \gamma^\mu u(p_1) p_1^{\nu}, \\
T_6^{\mu\nu} = \bar{u}(p_2) \gamma^\nu u(p_1) p_1^{\mu}, \\
T_7^{\mu\nu} = \bar{u}(p_2) \gamma^{\nu} u(p_1) p_1^{\mu}, \\
T_8^{\mu\nu} = \bar{u}(p_2) \gamma^{\rho\nu} u(p_1) p_2^{\mu}, \\
T_9^{\mu\nu} = \bar{u}(p_2) \gamma^{\rho} \gamma^{\nu} u(p_1), \\
T_{10}^{\mu\nu} = \bar{u}(p_2) \gamma^{\nu} \gamma^{\mu} u(p_1).
\]

$gg$ channel:

\[
T_1^{\mu\nu} = \epsilon_1 \cdot \epsilon_2 g^{\mu\nu}, \\
T_2^{\mu\nu} = \epsilon_1^\nu \epsilon_2^\mu, \\
T_3^{\mu\nu} = \epsilon_1^\nu \epsilon_2^\mu, \\
T_4^{\mu\nu} = \epsilon_1 \cdot \epsilon_2 p_1^\mu p_1^{\nu}, \\
T_5^{\mu\nu} = \epsilon_1 \cdot \epsilon_2 p_1^\mu p_2^{\nu}, \\
T_6^{\mu\nu} = \epsilon_1 \cdot \epsilon_2 p_2^\mu p_1^{\nu}, \\
T_7^{\mu\nu} = \epsilon_1 \cdot \epsilon_2 p_2^\mu p_2^{\nu}, \\
T_8^{\mu\nu} = \epsilon_2 \cdot p_3 \epsilon_1^\mu p_1^{\nu}, \\
T_9^{\mu\nu} = \epsilon_2 \cdot p_3 \epsilon_1^\mu p_2^{\nu}, \\
T_{10}^{\mu\nu} = \epsilon_2 \cdot p_3 \epsilon_1^\nu p_1^{\mu}, \\
T_{11}^{\mu\nu} = \epsilon_2 \cdot p_3 \epsilon_1^\nu p_2^{\mu}, \\
T_{12}^{\mu\nu} = \epsilon_1 \cdot p_3 \epsilon_2^\mu p_1^{\nu}, \\
T_{13}^{\mu\nu} = \epsilon_1 \cdot p_3 \epsilon_2^\mu p_2^{\nu}, \\
T_{14}^{\mu\nu} = \epsilon_1 \cdot p_3 \epsilon_2^\nu p_1^{\mu}, \\
T_{15}^{\mu\nu} = \epsilon_1 \cdot p_3 \epsilon_2^\nu p_2^{\mu}, \\
T_{16}^{\mu\nu} = \epsilon_1 \cdot p_3 \epsilon_2 \cdot p_3 g^{\mu\nu}, \\
T_{17}^{\mu\nu} = \epsilon_1 \cdot p_3 \epsilon_2 \cdot p_3 p_1^\mu p_1^{\nu}, \\
T_{18}^{\mu\nu} = \epsilon_1 \cdot p_3 \epsilon_2 \cdot p_3 p_1^\nu p_2^{\mu}, \\
T_{19}^{\mu\nu} = \epsilon_1 \cdot p_3 \epsilon_2 \cdot p_3 p_2^\mu p_1^{\nu}, \\
T_{20}^{\mu\nu} = \epsilon_1 \cdot p_3 \epsilon_2 \cdot p_3 p_2^\nu p_2^{\mu}.
\]
Helicity amplitudes for $q\bar{q}' \rightarrow V_1 V_2 \rightarrow l_5\bar{l}_6 l_7\bar{l}_8$

\[
M_{\lambda LL}^{V_1V_2}(p_1, p_2; p_5, p_6, p_7, p_8) = i (4\pi\alpha)^2 \sum_j \frac{L_{V_1}^{l_5l_6} L_{V_2}^{l_7l_8} Q_{q' q}}{D_{V_1}(p_3) D_{V_2}(p_4)} M_{\lambda LL}^{[j]}(p_1, p_2; p_5, p_6, p_7, p_8)
\]

where $M_{LLL}$ and $M_{RLL}$ independent, others given by crossing relations. E.g.:

\[
M_{LLL}(p_1, p_2; p_5, p_6, p_7, p_8) = [1 \rho/3 2] \left\{ E_1 \langle 15 \rangle \langle 17 \rangle [16][18] + E_2 \langle 15 \rangle \langle 27 \rangle [16][28] + E_3 \langle 25 \rangle \langle 17 \rangle [26][18] + E_4 \langle 25 \rangle \langle 27 \rangle [26][28] + E_5 \langle 57 \rangle [68] \right\} + E_6 \langle 15 \rangle \langle 27 \rangle [16][18] + E_7 \langle 25 \rangle \langle 27 \rangle [26][18] + E_8 \langle 25 \rangle \langle 17 \rangle [16][18] + E_9 \langle 25 \rangle \langle 27 \rangle [16][28],
\]

Only 9 out of 10 independent form factors relevant for $d = 4$:

\[
E_1 = A_1, \quad E_6 = 2 A_7 + \frac{2(u - p_3^2)}{s} (A_9 - A_{10}),
\]

\[
E_2 = A_2 + \frac{2}{s} (A_9 - A_{10}), \quad E_7 = 2 A_8 - \frac{2(t - p_3^2)}{s} (A_9 - A_{10}),
\]

\[
E_3 = A_3 - \frac{2}{s} (A_9 - A_{10}), \quad E_8 = 2 A_5 - \frac{2}{s} [(u - s - p_3^2)A_9 + (t - p_4^2)A_{10}],
\]

\[
E_4 = A_4, \quad E_9 = 2 A_6 - \frac{2}{s} [(t - s - p_3^2)A_{10} + (u - p_4^2)A_9],
\]

\[
E_5 = 2(A_9 + A_{10}),
\]

Andreas v. Manteuffel (Mainz)

Feynman integrals for QCD

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Reduze 2 [AvM, C. Studerus]
arXiv:1201.4330, HepForge

based on: Reduze [Studerus '09],
GiNaC [Bauer, Frink, Kreckel '00],
Fermat [Lewis]

- distributed Feynman integral reduction
- advanced shift finders
- upcoming version features:
  - **bilinear propagators**
    (3-loop heavy flavour Wilson coefficients in DIS [Blümlein et al. ‘13–’14])
  - **phase space integrals**
    (soft-virtual N^3LO Higgs and DY [Li, AvM, Schabinger, Zhu ‘14])
  - **finite integral finder + dimension shifts**
    (dims & dots method [AvM, Panzer, Schabinger ‘14])
  - family finder, ...
Master integrals for $q\bar{q}' \rightarrow VV'$ and $gg \rightarrow VV'$

84 master integrals (w/ products, w/o crossings)

planar two-loop master integrals

non-planar master integrals
An improved basis for differential equations

- method by [Kotikov '91]; [Gehrmann, Remiddi '99], relies on IBP reduction
- system of diff. eqns for basis integrals wrt external invariants \((\epsilon = (4 - d)/2)\):
  \[
  \frac{\partial}{\partial s_i} \mathbf{M}(\epsilon, s) = \mathbf{A}^{(s_i)}(\epsilon, s) \mathbf{M}(\epsilon, s)
  \]
- in certain cases proper choice of basis achieves [Kotikov '10]; [Henn '13]:
  \[
  \mathbf{A}^{(s_i)}(\epsilon, s) = \epsilon \mathbf{A}^{(s_i)}(s)
  \]
  such that

  \[
  d\mathbf{M}(\epsilon, s) = \epsilon \sum_n \mathbf{A}^{(n)} d\ln I_n(s) \mathbf{M}(\epsilon, s)
  \]
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  \]
  such that
  \[
  d\vec{M}(\epsilon, s) = \epsilon \sum_n \vec{A}^{(n)} d\ln l_n(s) \vec{M}(\epsilon, s)
  \]

features:
- full decoupling after expansion in \( \epsilon \):
  \[
  \vec{M} = \vec{M}^{(0)} + \epsilon \vec{M}^{(1)} + \ldots
  \]
  \[
  d\vec{M}^{(k)}(s) = \sum_n \vec{A}^{(n)} d\ln l_n(s) \vec{M}^{(k-1)}(s)
  \]
- every term of \( \epsilon \) expansion: multiple polylogs of uniform weight
- applies to phase space integrals [Höschele, Hoff, Ueda '14]; [AvM, Schabinger, Zhu '14]
- construction of canonical form: [Lee '14], see talk by [Tancredi]
- more applications: see talk by [Henn]
Master integrals for $q\bar{q}' \to VV'$ and $gg \to VV'$
STRUCTURE OF RESULT

vector of 111 master integrals in canonical basis with alphabet:

\[ \{ \bar{t}_1, \ldots, \bar{t}_{20} \} = \{ 2, \bar{x}, 1 + \bar{x}, 1 - \bar{y}, \bar{y}, 1 + \bar{y}, 1 - \bar{x}\bar{y}, 1 + \bar{x}\bar{y}, 1 - \bar{z}, \bar{z}, \\
1 + \bar{y} - 2\bar{y}\bar{z}, 1 - \bar{y} + 2\bar{y}\bar{z}, 1 + \bar{x}\bar{y} - 2\bar{x}\bar{y}\bar{z}, 1 - \bar{x}\bar{y} + 2\bar{x}\bar{y}\bar{z}, \\
1 + \bar{y} + \bar{x}\bar{y} + \bar{x}\bar{y}^2 - 2\bar{y}\bar{z} - 2\bar{x}\bar{y}\bar{z}, 1 + \bar{y} - \bar{x}\bar{y} - \bar{x}\bar{y}^2 - 2\bar{y}\bar{z} + 2\bar{x}\bar{y}\bar{z}, \\
1 - \bar{y} - \bar{x}\bar{y} + \bar{x}\bar{y}^2 + 2\bar{y}\bar{z} + 2\bar{x}\bar{y}\bar{z}, 1 - \bar{y} + \bar{x}\bar{y} - \bar{x}\bar{y}^2 + 2\bar{y}\bar{z} - 2\bar{x}\bar{y}\bar{z}, \\
1 - 2\bar{y} - \bar{x}\bar{y} + \bar{y}^2 + 2\bar{x}\bar{y}^2 - \bar{x}\bar{y}^3 + 4\bar{y}\bar{z} + 2\bar{x}\bar{y}\bar{z} + 2\bar{x}\bar{y}^3\bar{z}, \\
1 - \bar{y} - 2\bar{x}\bar{y} + 2\bar{x}\bar{y}^2 + \bar{x}^2\bar{y}^2 - \bar{x}^2\bar{y}^3 + 2\bar{y}\bar{z} + 4\bar{x}\bar{y}\bar{z} + 2\bar{x}^2\bar{y}^3\bar{z} \} \]

in parametrisation which rationalizes root of Källén function \( \sqrt{s^2 + p_3^4 + p_4^4 - 2(s p_3^2 + p_3^2 p_4^2 + p_4^2 s)} \):

\[ s = \bar{m}^2 (1 + \bar{x})^2, \quad t = -\bar{m}^2 \bar{x}((1 + \bar{y})(1 + \bar{x}\bar{y}) - 2\bar{z}\bar{y}(1 + \bar{x})), \quad p_3^2 = \bar{m}^2 \bar{x}^2 (1 - \bar{y}^2), \quad p_4^2 = \bar{m}^2 (1 - \bar{x}^2 \bar{y}^2) \]

integrated in terms of:

MULTIPLE POLYLOGARITHMS [Remiddi, Gehrmann]; [Goncharov]

\[ G(a_1, a_2, \ldots, a_n; x) = \int_0^x dt \frac{dt}{t - a_1} G(a_2, \ldots, a_n; t), \]

- independent input for a couple of very simple bubbles and triangles
- remaining boundary functions fixed by regularity
- checked against SecDec 2 [Borowka, Carter, Heinrich '12]
- symbol and more [Brown '11], [Duhr '12], [Duhr, Gangl, Rhodes '11], [Vollinga, Weinzierl '04]
\textbf{example result:} (dots are squared propagators)

in \textbf{traditional functional basis}:

\[
- \epsilon^2 \bar{m}^2 \epsilon t = 1 + \epsilon \left[ - 2G(-1, \bar{y}) - 2G(0, \bar{x}) - 2G(-1/\bar{y}, \bar{x}) - 2G(((1 + \bar{y})(1 + \bar{x}\bar{y})))/(2(1 + \bar{x})\bar{y}), \bar{z}) \right] \\
+ \epsilon^2 \left[ 4G(0, \bar{x})G(((1 + \bar{y})(1 + \bar{x}\bar{y})))/(2(1 + \bar{x})\bar{y}), \bar{z}) + 4G(-1/\bar{y}, \bar{x})G(((1 + \bar{y})(1 + \bar{x}\bar{y})))/(2(1 + \bar{x})\bar{y}), \bar{z}) \right] \\
+ G(-1, \bar{y})G(0, \bar{x}) + 4G(-1/\bar{y}, \bar{x}) + 4G(((1 + \bar{y})(1 + \bar{x}\bar{y})))/(2(1 + \bar{x})\bar{y}), \bar{z}) + 4G(-1, -1, \bar{y}) \\
+ 4G(0, 0, \bar{x}) + 4G(0, -1/\bar{y}, \bar{x}) + 4G(-1/\bar{y}, 0, \bar{x}) + 4G(-1/\bar{y}, -1/\bar{y}, \bar{x}) \\
+ 4G(((1 + \bar{y})(1 + \bar{x}\bar{y})))/(2(1 + \bar{x})\bar{y}), ((1 + \bar{y})(1 + \bar{x}\bar{y})))/(2(1 + \bar{x})\bar{y}), \bar{z}) \right] \\
+ O(\epsilon^3)
\]

in \textbf{optimized functional basis} for numerical evaluation:

\[
- \epsilon^2 m^2 \epsilon t = 1 + \epsilon \left[ - 2 \ln(l_1) - 2 \ln(l_5) \right] + \epsilon^2 \left[ 2 \ln^2(l_1) + 4 \ln(l_1) \ln(l_5) + 2 \ln^2(l_5) \right] + O(\epsilon^3)
\]
OPTIMISED FUNCTIONAL BASIS

choose real valued $\ln l_i$, $Li_n(R_1)$, $Li_{2,2}(R_1, R_2)$ with

$$|R_1| < 1, \quad |R_1 R_2| < 1$$

where $R_i$ are power products of letters (e.g. $-l_1, l_3, -l_8/(l_1 l_3), \ldots$)

such that $Li$ functions have convergent power series

$$Li_n(R_1) = -\sum_{j_1=1}^{\infty} \frac{R_1^{j_1}}{j_1^n}, \quad Li_{2,2}(R_1, R_2) = \sum_{j_1=1}^{\infty} \sum_{j_2=1}^{\infty} \frac{R_1^{j_1}}{(j_1 + j_2)^2} \frac{(R_1 R_2)^{j_2}}{j_2^2}$$

features:

- symbol based rewriting [Goncharov, Spradlin, Vergu, Volovich '10]
- algorithmic argument construction [Duhr, Gangl, Rhodes '11]
- require absence of spurious letters
- fast and stable numerical evaluation:
  - $O(150\text{ms})$ full off-shell helicity amplitudes
  - $O(35\text{ms})$ equal mass interferences
  - orders of magnitude faster than traditional representation
Symbol based integration

- DGR basis: circumvents artificial linearisation of alphabet

- here: new parametrisation $s = m^2 (1 + x)(1 + xy)$, $t = -m^2 xz$, $p_3^2 = m^2$, $p_4^2 = m^2 x^2 y$:

  \[ \{l_1, \ldots, l_{17}\} = \{x, 1 + x, y, 1 - y, z, 1 - z, -y + z, 1 + y - z, 1 + xy, 1 + xz, xy + z, 1 + y + xy - z, 1 + x + xy - xz, 1 + y + 2xy - z + x^2 yz, 2xy + x^2 y + x^2 y^2 + z - x^2 yz, 1 + x + y + xy + xy^2 - z - xz - xyz, 1 + y + xy + y^2 + xy^2 - z - yz - xyz\} \]

  very non-linear, but shorter than previous alphabet

- skip “traditional integration” and “integrate symbol”:

  \[
  \begin{align*}
  dM_k &= A^{(n)} M_{k-1} \, d\ln l_n \\
  S(M_k) &= A^{(n)} S(M_{k-1}) \otimes l_n
  \end{align*}
  \]

in $N = 4$: [Dixon, Drummond, Duhr, Pennington ’14]
in SM Drell-Yan production: [AvM, Schabinger (to appear)]
helicity amplitudes for $q\bar{q}' \rightarrow VV'$ @ 2-loops [Gehrmann, AvM, Tancredi '15]
RESULT: $W^+W^-$ production at NNLO

NNLO prediction significantly reduces tension with data:

![Graph showing $W^+W^-$ production at different orders of approximation with CMS and ATLAS data points.]

[Gehrmann, Grazzini, Kallweit, Maierhöfer, AvM, Pozzorini, Rathlev, Tancredi ‘14]
VVamp project

This is the web page of the Vamp project. We provide the two-loop helicity amplitudes for electroweak vector boson pair production and their decay into 4 leptons in quark-antiquark annihilation and in gluon-gluon fusion.

You can download our analytical results for the master integrals and the amplitudes. Moreover, we provide C++ implementations for the fast and reliable numerical evaluation of the amplitudes.

Reference


Downloads: amplitudes

- bare form factors exact in d: class A, class B, class C (Form format)
- finite form factors in qt-scheme: class A, class B, class C (Form format)
- relations for projectors: Ai, taui of Ai (Form format)
- numerical implementation of form factors: qvamp package (C++, requires GiNaC)

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Downloads: master integrals

- master integral definitions: Mathematica, Form format
- master integral traditional solutions: Mathematica, Form format
- master integral optimised solutions: Mathematica, Form format
- master integral crossing relations: Mathematica, Form format
- integral families, kinematics (in Reduze 2 format)
Part II: A basis of finite Feynman integrals

[AvM, Panzer, Schabinger]
An improved basis for Feynman parameters

consider **Feynman parameter representation** of multi-loop integral

\[
I = \frac{\Gamma(\nu - \frac{Ld}{2})(-1)\nu}{\prod_{i=1}^{N} \Gamma(\nu_i)} \left[ \prod_{j=1}^{N} \int_{0}^{\infty} dx_j \right] \delta(1 - x_N) \mathcal{U}^{\nu-(L+1)d/2} \mathcal{F}^{\nu+Ld/2} \prod_{k=1}^{N} x_k^{\nu_k-1}
\]

where \( \nu = \sum_i \nu_i \), \( \nu_i \) denotes propagator multiplicity

presence of **subdivergencies** (= divergencies from Feynman parameter integrations) implies:
- can’t directly expand in \( \epsilon = (4 - d)/2 \)
- no straight-forward analytical integration a la [Brown ’08; Panzer ’14]
- no straight-forward numerical integration

generic approaches to **singularity resolution**:
- sector decomposition [Binoth, Heinrich ’00], see talk by [Borowka]
- polynomial exponent raising [Tkachov ’96, Passarino ’00]
- regularising dimension shifts [Panzer ’14]
- basis of finite Feynman integrals [AvM, Schabinger, Panzer ’14]
Sector decomposition: shortcomings
calculate to $O(\epsilon)$:

$$I(\epsilon) = \int_0^1 dt \ t^{-1-\epsilon}(1-t)^{-1-2\epsilon} \ _2F_1(\epsilon, 1-\epsilon; -\epsilon; t)$$

decompose into sectors: split at (arbitrary) $t = 1/2$:

$$I_1(\epsilon) = \int_0^{1/2} dt \ t^{-1-\epsilon}(1-t)^{-1-2\epsilon} \ _2F_1(\epsilon, 1-\epsilon; -\epsilon; t)$$

$$I_2(\epsilon) = \int_{1/2}^1 dt \ t^{-1-\epsilon}(1-t)^{-1-2\epsilon} \ _2F_1(\epsilon, 1-\epsilon; -\epsilon; t) .$$

rescale, expand in plus distributions, evaluate:

$$I_1(\epsilon) = -\frac{1}{\epsilon} - 1 + \left(3 + \frac{1}{3} \pi^2 - 8 \ln(2)\right) \epsilon + O(\epsilon^2)$$

$$I_2(\epsilon) = -\frac{1}{3\epsilon} + \frac{7}{3} + \left(-7 + \frac{1}{3} \pi^2 + 8 \ln(2)\right) \epsilon + O(\epsilon^2) .$$

result:

$$I(\epsilon) = -\frac{4}{3\epsilon} + \frac{4}{3} + \left(-4 + \frac{2}{3} \pi^2\right) \epsilon + O(\epsilon^2) .$$

note:

- split up of domain introduces spurious terms $\ln(2)$
- spurious order 5 polynomial denominators: [AvM, Schabinger, Zhu '13]
- destroys linear reducibility & prevents analytical integration
An example for subdivergencies

\[
\begin{align*}
\mathcal{U} &= \int \frac{d^d k_1}{i \pi^{d/2}} \int \frac{d^d k_2}{i \pi^{d/2}} \frac{1}{((k_1 + k_2)^2 - m^2) k_1^2 k_2^2} \\
&= -\Gamma(-1 + 2\epsilon) \int_0^\infty dx_1 \delta(1 - x_1) \int_0^\infty dx_2 \int_0^\infty dx_3 \mathcal{U}^{-3 + 3\epsilon} \mathcal{F}^{1 - 2\epsilon},
\end{align*}
\]

with Symanzik polynomials

\[
\mathcal{U} = x_1 x_2 + x_1 x_3 + x_2 x_3 \quad \text{and} \quad \mathcal{F} = m^2 x_1 \mathcal{U}.
\]

- can't expand integrand in \( \epsilon \):

\[
\begin{align*}
\mathcal{U} &= \int \frac{d^d k_1}{i \pi^{d/2}} \int \frac{d^d k_2}{i \pi^{d/2}} \frac{1}{((k_1 + k_2)^2 - m^2) k_1^2 k_2^2} \\
&= - (m^2)^{1 - 2\epsilon} \frac{\Gamma(-1 + 2\epsilon)\Gamma(\epsilon)\Gamma(1 - \epsilon)}{1 - \epsilon}
\end{align*}
\]

\( \Gamma(\epsilon) \) signals subdivergence

- Euclidean integrals: all divergencies from integration boundaries

- notation here: restrict to one or several parameters approaching zero (not infinity)
**Systematic recognition of subdivergencies**

- follow [Panzer '14]
- consider subsets
  \[ \{x_1, x_2\}, \{x_1, x_3\}, \{x_2, x_3\}, \{x_1\}, \{x_2\}, \{x_3\} \]
- for each subset \( J \) consider scaling with \( \lambda \):
  \[ J \to \lambda J \]

for integrand \( P \equiv \mathcal{U}^{-3 + 3\epsilon} \mathcal{F}^{1 - 2\epsilon} \):

\[ P \to P\lambda = \lambda^{\deg J(P)} \tilde{P} \quad \text{where} \quad \lim_{\lambda \to 0} \tilde{P} = \mathcal{O}(\lambda^0) \]

and the integral measure

\[ \prod_{i=1}^{3} \prod_{i=1}^{3} dx_i \to \lambda^{|J|} \prod_{i=1}^{3} dx_i \]

and read off:

**convergence index**

\[ \omega_J(P) = |J| + \deg J(P), \]

\[ \lim_{\epsilon \to 0} \omega_J(P) \leq 0 \iff \text{presence of non-integrable subdvergence} \]
Panzer’s regularising shift

integrand can be regularized by dimension-shifts [Panzer '14]:

1. pick $J$ for which $\lim_{\epsilon \to 0} \omega_J(P) \leq 0$
2. multiply by $1 = \int_{0}^{\infty} d\lambda \, \delta(\lambda - x_J)$ with $x_J = \sum_{j \in J} x_j$
3. rescale $x_j \to \lambda x_j$ for all $j \in J$ and perform partial integration (surface term vanishes)
4. new integrand

$$P' = -\frac{1}{\omega_J(P)} \frac{\partial}{\partial \lambda} \tilde{P} \bigg|_{\lambda \to 1}.$$ 

has improved convergence by design

5. iterate until no subdivergencies
Panzer’s regularising shift

The integrand can be regularized by dimension-shifts [Panzer '14]:

1. Pick $J$ for which $\lim_{\epsilon \to 0} \omega_J(P) \leq 0$
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Has improved convergence by design
5. Iterate until no subdivergencies

Applicability in practice:

- Problem: proliferation of terms
- Solution: integration by parts (IBP) reductions
Our proposal: minimal dims & dots

decompose wrt quasi-finite basis

\[
\begin{align*}
(4-2\epsilon) &= \frac{4(1-\epsilon)(3-4\epsilon)(1-4\epsilon)}{\epsilon s^2} \\
&\quad - \frac{10 - 65\epsilon + 131\epsilon^2 - 74\epsilon^3}{\epsilon^3 s^2} \\
&\quad - \frac{14 - 119\epsilon + 355\epsilon^2 - 420\epsilon^3 + 172\epsilon^4}{(1-2\epsilon)\epsilon^3 s^3} \\
&\quad (4-2\epsilon)
\end{align*}
\]

basis consists of standard Feynman integrals, but

- in **shifted dimensions**
- with additional **dots** (propagators taken to higher powers)
- old reg. shifts generated $O(10\text{MB})$, here: 3 lines! (more severe at higher loops)
Existence of quasi-finite basis

1. start with some basis $B$ for topology and subtopologies
2. assume master $b$ not quasi-finite and has integrand

$$P = \mathcal{U}^{-(L+1)d/2} \mathcal{F}^{-\nu+Ld/2} \prod_{j=1}^{N} x_{j}^{\nu_{j}-1}, \quad \text{where } \nu = \sum_{i=1}^{N} \nu_{i}$$

3. consider regularizing dimension shift:

$$P' = -\frac{1}{\omega_{J}(P)} \prod_{j=1}^{N} x_{j}^{\nu_{j}-1} \left\{ \left( \nu - \frac{(L+1)d}{2} \right) \mathcal{U}^{(\nu+L)-(L+1)(d+2)/2} \mathcal{F}^{-(\nu+L)+L(d+2)/2} \frac{\partial \tilde{\mathcal{U}}}{\partial \lambda} \bigg|_{\lambda \rightarrow 1} \right\} + \mathcal{F} \text{ derivative term} \right\},$$

with $\mathcal{U}_{J,\lambda} = \lambda^{\deg_{J}(\mathcal{U})} \tilde{\mathcal{U}}$

4. picking any monomial from $\frac{\partial \tilde{\mathcal{U}}}{\partial \lambda} \bigg|_{\lambda \rightarrow 1}$ or $\frac{\partial \tilde{\mathcal{F}}}{\partial \lambda} \bigg|_{\lambda \rightarrow 1}$ gives

**dimension-shifted** and **dotted** integral with **improved convergence**!

5. choose one term such that new integral $b'$ is independent of $B \setminus b$
6. replace $b \rightarrow b'$ and iterate until $B$ free of subdivergences (quasi-finite)
7. optional: transition quasi-finite $\rightarrow$ finite integrals
Practical algorithm for basis construction

given the existence proof, forget about previous construction and just do:

**Algorithm: construction of (quasi-)finite basis**

- systematic scan for (quasi-)finite integrals with dim-shifts and dots
- IBP + dimensional recurrence for actual basis change

**Remarks:**
- computationally expensive part shifted to IBP solver (Fire, Reduze, LiteRed)
- efficient, easy to automate (implemented in dev. version of Reduze 2)
- any dim-shift good, e.g. shifts by [Tarasov '96], [Lee '10]
- see [Bern, Dixon, Kosower '93] for dim-shifted one-loop pentagon
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Example 1a: non-planar two-loop vertex (quasi-finite)

\[(4 - 2\epsilon)\]

\[\begin{align*}
(4 - 2\epsilon) &= (4 - 2\epsilon), \\
(4 - 2\epsilon) &= 2 - 3\epsilon \\
(4 - 2\epsilon) &= 4(1 - \epsilon)(3 - 4\epsilon)(1 - 4\epsilon) \\
& \quad \times \frac{\epsilon s^2}{\epsilon s^2} \\
& \quad - \frac{10 - 65\epsilon + 131\epsilon^2 - 74\epsilon^3}{\epsilon^3 s^2} \\
& \quad - \frac{14 - 119\epsilon + 355\epsilon^2 - 420\epsilon^3 + 172\epsilon^4}{(1 - 2\epsilon)\epsilon^3 s^3}.
\end{align*}\]
**Example 1B: Non-planar Two-Loop Vertex (Finite)**

\[
\begin{align*}
(4-2\epsilon) & = -\frac{4s^2}{\epsilon(1-2\epsilon)}, \\
(4-2\epsilon) & = \frac{2(2-3\epsilon)s}{\epsilon^2}, \\
(4-2\epsilon) & = -\frac{4(1-\epsilon)(1-4\epsilon)}{\epsilon s}, \\
& \quad -\frac{2(2-3\epsilon)(5-21\epsilon + 14\epsilon^2)}{\epsilon^4 s}, \\
& \quad +\frac{4(2-3\epsilon)(7-31\epsilon + 26\epsilon^2)}{\epsilon^4(1-2\epsilon)s}.
\end{align*}
\]
**Example 2: Massless Planar Double Box Family**

\[
\begin{align*}
  b_1 &= (6 - 2\epsilon) \\
  b_2 &= (6 - 2\epsilon) \\
  b_3 &= (6 - 2\epsilon) \\
  b_4 &= (6 - 2\epsilon) \\
  b_5 &= (6 - 2\epsilon) \\
  b_6 &= (4 - 2\epsilon) \\
  b_7 &= (4 - 2\epsilon) \\
  b_8 &= (6 - 2\epsilon)
\end{align*}
\]
Example 3: Three-loop form factor

- massless quark and gluon form factors:
  - simplest objects to study IR properties of QCD

- master integrals:
  - [Gehrmann, Heinrich, Huber, Studerus '06]
  - [Heinrich, Huber, Maître '07]
  - [Heinrich, Huber, Kosower, V. Smirnov '09]
  - [Lee, A. Smirnov, V. Smirnov '10]
  - [Baikov, Chetyrkin, A. Smirnov, V. Smirnov, Steinhauser '09]
  - [Lee, V. Smirnov '10] ⇔ the only complete weight 8
  - [Henn, A. Smirnov, V. Smirnov '14] (diff. eqns.)

- form factor @ 3-loops:
  - [Baikov, Chetyrkin, A. Smirnov, V. Smirnov, Steinhauser '09]
  - [Gehrmann, Glover, Huber, Ikizlerli, Studerus '10, '10]
Example 3: Three-loop form factor [AvM, Panzer, Schabinger; to appear]

\[ F_3^q = \frac{1}{\epsilon^6} \left[ c_1 + (10-2\epsilon) + c_2 + (8-2\epsilon) + c_3 + (10-2\epsilon) + c_4 + (6-2\epsilon) + c_5 + (10-2\epsilon) + c_6 + (10-2\epsilon) + c_7 + (8-2\epsilon) + c_8 + (6-2\epsilon) + \frac{1}{\epsilon^4} c_9 + (6-2\epsilon) \right] + \frac{1}{\epsilon^3} \left[ c_{10} + (6-2\epsilon) + c_{11} + (6-2\epsilon) + c_{12} + (8-2\epsilon) + c_{13} + (8-2\epsilon) + c_{14} + (6-2\epsilon) \right] + \frac{1}{\epsilon^2} \left[ c_{15} + (8-2\epsilon) + c_{16} + (6-2\epsilon) + \frac{1}{\epsilon^1} c_{17} + (6-2\epsilon) + c_{18} + (6-2\epsilon) \right] + \frac{1}{\epsilon} \left[ c_{19} + (6-2\epsilon) + c_{20} + (4-2\epsilon) + c_{21} + (4-2\epsilon) + c_{22} + (6-2\epsilon) \right] \]
Example 4: Four-loop form factor [AvM, Panzer, Schabinger; in progress]

- example: a non-planar 12-line top level topology @ 4-loops

- analytical result with HypInt [Panzer]:

\[
(6-2\epsilon) \approx 3.18074 + \mathcal{O}(\epsilon)
\]

- numerical result with Fiesta [A. Smirnov]:

\[
3.18082 + \epsilon 58.8288 + \mathcal{O}(\epsilon^2)
\]
Numerical evaluations

Advantages of (quasi-)finite basis:
- Straight-forward to integrate numerically (in principle)
- No blow up in number of numerical integrations (speed, stability)
- No cancellation of spurious structures (stability)

Experiments with numerical evaluations:
- Naive straight-forward implementation works already quite well
- Convenient: employ existing sector decomposition programs
  Fiesta, SecDec and sector decomposition
- (Quasi-)finite integrals: faster & more reliable
Conclusions

**differential equations:**
- powerful analytical method for multiscale integrals in QCD
- refinement via normal form basis (if applicable)
- essential: systematic treatment of multipole polylogs (symbol etc)
- optimize functional basis for result
- NNLO prediction for diboson production at LHC

**basis of finite integrals:**
- simple and efficient method for singularity resolution in multi-loop integrals
- analytical integrations: quasi-finite integrals are Feynman integrals (dim-shifted, dotted)
- numerical integrations: faster and more stable evaluations
- application: massless QCD form factors