Improve Numerical Stabilities of Amplitudes in Precise QCD Calculations

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Motivation of Precise QCD Calculations

- Precise QCD calculations involves:
  - PDF sets calculated and fitted with higher order splitting function.
  - Fixed order pQCD calculations including more loops and/or legs.
  - Parton shower, resummation etc.

- Motivations:
  - Reduced theoretical uncertainty
  - Large contributions from higher order terms in pQCD
  - Better understanding of S/B in LHC
  - Distinguish SM signal from potential new physics

Example of cutting edge studies:

- $pp \rightarrow H @ 3$ LO
  - Anastasiou et al

- Di-jet production @ NNLO
  - Currie et al

- Top pair production @ NNLO
  - Abelof et al; Baernreuther et al

- $H + jet @ NNLO$
  - Chen et al; Boughezal et al

- $Z + jet @ NNLO$
  - Morgan et al

- $W + jet @ NNLO$
  - Boughezal et al

- Higgs and Drell-Yan production @ NNLO + PS
  - Hamilton et al

- Colourless particles production @ NNLO + NNLL
  - Wiesemann et al

And many more
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- Example of cutting edge studies:
  - $pp \rightarrow H \oplus N^3LO$ Anastasiou et al
  - Di-jet production $\oplus NNLO$ Currie et al
  - top pair production $\oplus NNLO$ Abelof et al; Baernreuther et al
  - H+jet $\oplus NNLO$ Chen et al; Boughezal et al
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  - And many more ...
Matrix elements involved in fixed order pQCD

- Renormalised factorized parton level differential cross section ($d\hat{\sigma}$) for example:

$$d\hat{\sigma}_{LO} = \int [\langle M^0 | M^0 \rangle]_{n+2} d\Phi_n J_n^{(n)}$$

$$d\hat{\sigma}_{NLO} = \int [\langle M^0 | M^0 \rangle]_{n+3} d\Phi_{n+1} J_{n}^{(n+1)}$$

$$+ \int [\langle M^0 | M^1 \rangle + \langle M^1 | M^0 \rangle]_{n+2} d\Phi_n J_n^{(n)}$$

$$d\hat{\sigma}_{NNLO} = \int [\langle M^0 | M^0 \rangle]_{n+4} d\Phi_{n+2} J_n^{(n+2)}$$

$$+ \int [\langle M^0 | M^1 \rangle + \langle M^1 | M^0 \rangle]_{n+3} d\Phi_{n+1} J_{n}^{(n+1)}$$

$$+ \int [\langle M^1 | M^1 \rangle + \langle M^2 | M^0 \rangle + \langle M^0 | M^2 \rangle]_{n+2} d\Phi_n J_n^{(n)}$$

- Higher order contributions contains both explicit (Pole structure) and implicit IR divergences (singular in unresolved P.S.).
Matrix elements involved in fixed order pQCD

- Renormalised factorized parton level differential cross section ($d\hat{\sigma}$) for example:

\[
\begin{align*}
\hat{\sigma}_{LO} &= \int [\langle M^0|M^0\rangle]_{n+2} d\Phi_n J_n^{(n)} \\
\hat{\sigma}_{NLO} &= \int [\langle M^0|M^0\rangle]_{n+3} d\Phi_{n+1} J_n^{(n+1)} \\
&\quad + \int [\langle M^0|M^1\rangle + \langle M^1|M^0\rangle]_{n+2} d\Phi_n J_n^{(n)} \\
\hat{\sigma}_{NNLO} &= \int [\langle M^0|M^0\rangle]_{n+4} d\Phi_{n+2} J_n^{(n+2)} \\
&\quad + \int [\langle M^0|M^1\rangle + \langle M^1|M^0\rangle]_{n+3} d\Phi_{n+1} J_n^{(n+1)} \\
&\quad + \int [\langle M^1|M^1\rangle + \langle M^2|M^0\rangle + \langle M^0|M^2\rangle]_{n+2} d\Phi_n J_n^{(n)}
\end{align*}
\]

- Higher order contributions contains both explicit (Pole structure) and implicit IR divergences (singular in unresolved P.S.).
- Whether those matrix elements are stable in unresolved P.S. is an open question.
Factorisation of implicit IR divergence (NNLO)

- Implicit IR divergent behaviour of qQCD matrix elements can be factorised.
- Colour ordered amplitudes constrain the IR divergence only in colour connected partons Mangano, Parke, Giele, Xu, Berends (1980s)
- For single unresolved limits (tree level): Define $|M^0|^2 \equiv \langle M^0 | M^0 \rangle$

$$|M^0(\cdots, i, j_g, k, \cdots)|^2 \xrightarrow{p_j \to 0 \sim \Delta^2} S_{ijk} |M^0(\cdots, i, k, \cdots)|^2 \sim \mathcal{O}(\Delta^{-4})$$

$$|M^0(\cdots, i, j, \cdots)|^2 \xrightarrow{p_i//p_j} \frac{1}{s_{ij}} P_{ij \to K(z)} |M^0(\cdots, K, \cdots)|^2 \sim \mathcal{O}(\Delta^{-2})$$

where $s_{ij} = (p_i + p_j)^2$, $z = p_j / (p_j + p_i)$

$$S_{ijk} = \frac{2s_{ik}}{s_{ij}s_{jk}}$$

$$P_{qg \to Q} = P_{qg \to \bar{Q}} = \frac{1 + (1 - z)^2 - \epsilon z^2}{z}$$

$$P_{q\bar{q} \to G} = P_{q\bar{q} \to G} = \frac{z^2 + (1 - z)^2 - \epsilon}{1 - \epsilon}$$

$$P_{gg \to G} = 2 \left( \frac{z}{1 - z} + \frac{1 - z}{z} + z(1 - z) \right)$$
Factorisation of implicit IR divergence (NNLO)

- For single unresolved limits (loop level): Define
  \[ |M^1|^2 \equiv \langle M^0 | M^1 \rangle + \langle M^1 | M^0 \rangle \]
  \[
  |M^1(\cdots, i, j_g, k, \cdots)|^2 \xrightarrow{p_j \text{ soft}} S_{ijk} |M^1(\cdots, i, k, \cdots)|^2 \\
  + S^1_{ijk} |M^0(\cdots, i, k, \cdots)|^2 \sim \mathcal{O}(\Delta^{-4})
  \]
  \[
  |M^1(\cdots, i, j, \cdots)|^2 \xrightarrow{p_i \parallel p_j} \frac{1}{s_{ij}} P_{ij \rightarrow K(z)} |M^1(\cdots, K, \cdots)|^2 \\
  + \frac{1}{s_{ij}} P^1_{ij \rightarrow K(z)} |M^0(\cdots, K, \cdots)|^2 \sim \mathcal{O}(\Delta^{-2})
  \]

- For double unresolved limits (tree level):
  \[
  |M^0(\cdots, a, i, j, b, \cdots)|^2 \xrightarrow{p_i, p_j \text{ soft}} S_{aijb} |M^0(\cdots, a, b, \cdots)|^2 \sim \mathcal{O}(\Delta^{-8})
  \]
  \[
  |M^0(\cdots, i, j, k, \cdots)|^2 \xrightarrow{p_i \parallel p_j \parallel p_k} P_{ijk \rightarrow A(z_{1,2,3})} |M^0(\cdots, A, \cdots)|^2 \sim \mathcal{O}(\Delta^{-6})
  \]
  \[
  |M^0(\cdots, a, i, j, k, \cdots)|^2 \xrightarrow{p_i \text{ soft}, p_j \parallel p_k} \frac{1}{s_{jk}} S_{a,ijk} P_{jk \rightarrow K(z)} |M^0(\cdots, a, K, \cdots)|^2
  \]
Testing numerical stability of matrix elements

- Numerical instability comes from internal cancellation of terms with divergent order higher than the factorisation functions.
- Analytically check in each unresolved limits with known factorisation functions
  - Keep tracking of the order of divergences and find large cancellation behaviour
  - Only easy to check with small number of legs
  - Matrix elements calculated using different methods needs independent test

\[ |M_{0n+4}(\ldots, i, j, k, l\ldots)\rceil^2 \rightarrow X_{04}(i, j, k, l) \rceil M_{0n+2}(\ldots, I, L\ldots)\rceil^2 \]
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  - Matrix elements calculated using different methods needs independent test
- Numerically check with known factorisation functions in unresolved P.S. point
  - Can use each factorisation functions for comparison
  - Hard to relate parameters in exactly limit with unresolved P.S. points
  - Can also use special functions that converge to different factorisation functions
  - Hard to construct by hand
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  - Can also use special functions that converge to different factorisation functions
  - Hard to construct by hand
- Solution is to combine analytical and numerical methods
  - Analytically check matrix elements against factorisation functions
  - Use analytical checked matrix elements to construct special functions for numerical check (Antenna functions)
  - Recycle $|M_{n+4}^0(\cdots, i, j, k, l \cdots)|^2 \rightarrow X_4^0(i, j, k, l)|M_{n+2}^0(\cdots, I, L, \cdots)|^2$
Antenna functions: multi-purpose factorisation functions

Gehrmann-De Ridder, Gehrmann, Glover

- Antenna functions constructed from normalised matrix elements
- Each function has two specified hard radiators + 1 or 2 unresolved patrons

\[
X_3^0(i, j, k) \sim \frac{|M_{ijk}^0|^2}{|M_{IL}^0|^2}
\]

\[
X_3^1(i, j, k) \sim \frac{|M_{ijk}^1|^2}{|M_{IK}^0|^2} - X_{ijk}^0 \frac{|M_{IK}^1|^2}{|M_{IK}^0|^2}
\]

\[
X_4^0(i, j, k, l) \sim \frac{|M_{ijkl}^0|^2}{|M_{IL}^0|^2}
\]

- One antenna function mimics multiple double or single unresolved behaviour.
- Antenna functions calculated from different ME for all parton combinations:

\[
\gamma^* \rightarrow q\bar{q} + \text{partons} \quad \tilde{\chi} \rightarrow \tilde{g} + \text{partons} \quad H \rightarrow \text{partons}
\]

- Momentum mappings give the P.S. for reduced ME:

3 \rightarrow 2 \text{ or } 4 \rightarrow 2 \text{ mapping } \otimes \{FF, IF, II\} \text{ combinations of hard radiators.}
Testing numerical stability of matrix elements

- Construct antenna subtraction terms (ATS) to mimic unresolved limits of ME
  \[ ME^0 = |M^0(\cdots, i, j, k, \cdots)|^2, \quad ATS^0 = X_3^0(i, j, k)|M^0(\cdots, I, K, \cdots)|^2 \]
  \[ ME^1 = |M^1(\cdots, i, j, k, \cdots)|^2, \quad ATS^1 = X_3^0(i, j, k)|M^1(\cdots, I, K, \cdots)|^2 + X_3^1(i, j, k)|M^0(\cdots, I, K, \cdots)|^2 \]

- Test structure
  \[ R = \frac{ME^{0,1}}{ATS^{0,1}} \]

- \( R \sim \)horizontal axis (centre at one near the unresolved region)

- Number of P.S. points in each bin \( \sim \)vertical axis

- Controlling singular region correctly will achieve spike plot. For example:
  \[ x = \frac{s_{45}}{s}, \quad xS = \sum_{i \neq 3} s_{3i} \]
Testing numerical stability of matrix elements

- Numerical stabilities are tested for the following ME for NNLO studies:
  - Tree level: $M^0_\gamma (5P), M^0_0 (6P), M^0_H (5P) \ (EFT), M^0_Z (5P), M^0_W (5P)$
  - Loop level: $M^1_\gamma (4P), M^1_0 (5P), M^1_H (4P) \ (EFT), M^1_Z (4P), M^1_W (4P)$
  - Generate unresolved P.S. points and test all possible limits

Identify the unstable part of the amplitude (find NAN; change precision of variable; find identical large values)
Testing numerical stability of matrix elements

- Numerical stabilities are tested for the following ME for NNLO studies:
  - Tree level: $\mathcal{M}_0^\gamma(5P), \mathcal{M}_0^0(6P), \mathcal{M}_0^\gamma(5P) (EFT), \mathcal{M}_0^Z(5P), \mathcal{M}_0^W(5P)$
  - Loop level: $\mathcal{M}_1^\gamma(4P), \mathcal{M}_1^0(5P), \mathcal{M}_1^\gamma(4P) (EFT), \mathcal{M}_1^Z(4P), \mathcal{M}_1^W(4P)$
  - Generate unresolved P.S. points and test all possible limits
- Abnormal spike plots are found for single soft limits in $\mathcal{M}_1^\gamma(4P) (EFT)$:

Figure: $|M_H^1(gggg)|^2 \quad x_s = \sum_{i \neq 4} s_{4i}$

Figure: $|M_H^1(qgg\bar{q})|^2 \quad x_s = \sum_{i \neq 3} s_{3i}$

- Identify the unstable part of the amplitude (find NAN; change precision of variable; find identical large values)
Amplitudes for $\mathcal{M}_H^1(4P)$

- $\mathcal{M}_H^1(4P)$ (EFT) are calculated in hep-ph:0909.4457 and implemented in MCFM:
  - Use generalised unitarity method to construct the cut-constructible contributions.
  - A hybrid of Feynman diagram and recursive based techniques to determine the rational piece.
  - All partons are considered massless and the Higgs boson only couples to $g$ (EFT).

- The general structure is:

$$\mathcal{M}_H^1(4P) = C_4(4P) + R_4(4P)$$

$$C_4(4P) = \sum_i C_{4;i} I_{4;i} + \sum_i C_{3;i} I_{3;i} + \sum_i C_{2;i} I_{2;i}$$

- $I_{j;i}$ represents a j-point scalar basis integral (box, triangle, bubble)
- $C_{j;i}$ coefficients of basis integrals are calculated by on shell tree amplitudes
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  - Numerical instabilities in $\mathcal{M}_H^1(4P)$ come from NMHV amplitude $\mathcal{M}_H^1(1^+, 2^-, 3^-, 4^-)$:
    - $C_{3;1234|1234}$ coefficient of three mass triangle integral
    - Large cancellation of terms between $C_4(4P)$ and $R_4(4P)$
Rewrite three-mass coefficients for $\mathcal{M}^1_H(4P)$

The finite contributions of cut-constructible contributions contain:

$$C_4(1^+, 2^-, 3^-, 4^-) \sim C_{3;1234|12|34}(H, 1^+, 2^-, 3^-, 4^-) I_{3,3m}(m_H^2, s_{12}, s_{34})$$

with $(s_{ij} = \langle ij \rangle[ji])$

$$C_{3;1234|12|34}(H, 1^+, 2^-, 3^-, 4^-) = \sum_{\gamma=\gamma\pm} \frac{m_H^4 \langle 34 \rangle^3 \langle 2|K_1^b|1\rangle \langle 2|K_1^b|3\rangle \langle 2|K_1^b|4\rangle}{2\gamma(\gamma + m_H^2) \langle 12 \rangle s_1 K_1^b s_3 K_1^b s_4 K_1^b}$$

$$K_1^{\mu} = \gamma \frac{K_1^\mu - K_1^2 K_2^\mu}{\gamma^2 - K_1^2 K_2^2}$$

where $K_1$, $K_2$ (and $K_3$) are the momenta of the three off-shell legs and where $\gamma$ is determined by the two solutions that ensure that $K_1^b$ is light-like

$$K_1^\mu = -p_1^\mu - p_2^\mu - p_3^\mu - p_4^\mu, \quad K_2^\mu = p_1^\mu + p_2^\mu, \quad K_3^\mu = p_3^\mu + p_4^\mu$$

$$\gamma^2 - 2K_1 \cdot K_2 \gamma + K_1^2 K_2^2 = 0$$

$$\gamma_{\pm} = K_1 \cdot K_2 \pm \sqrt{(K_1 \cdot K_2)^2 - K_1^2 K_2^2}$$
Rewrite three-mass coefficients for $M_{H}^{1}(4P)$

- Solutions of $\gamma$ satisfy following identities:

  \[
  \gamma_+ + \gamma_- = 2K_1 \cdot K_2, \quad \gamma_+ \gamma_- = K_1^2 K_2^2, \\
  (\gamma_- + K_1^2)(\gamma_+ + K_1^2) = K_1^2 K_3^2, \\
  (\gamma_- + K_2^2)(\gamma_+ + K_2^2) = K_2^2 K_3^2
  \]
Rewrite three-mass coefficients for $\mathcal{M}^1_H(4P)$

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  (\gamma_- + K_2^2)(\gamma_+ + K_2^2) = K_2^2 K_3^2
  \]

- In general $C_{3;K_1|K_2|K_3}$ is sensitive to the three massive momentum inputs $K_1$, $K_2$ and $K_3$ when one of the legs becomes massless (e.g. $K_3^2 \to 0$):
  \[
  -2K_1 \cdot K_2 \to K_1^2 + K_2^2 \quad \gamma_+ \to -K_2^2 \quad \gamma_- \to -K_1^2
  \]

- A potentially large cancellation for example is inside $s_{3K_1^b}$:
  \[
  s_{3K_1^b} = -\frac{-\gamma_- (K_1^2 + \gamma_-)(s_{13} + s_{23}) - \gamma_-^2 s_{34}}{\gamma_-^2 - K_1^2 K_2^2},
  \]

- The result of $(K_1^2 + \gamma_-)$ in $K_3^2 \to 0$ limit is analytically proportional to the small value $K_3^2 = s_{34}$.

- Numerically unstable when the result of large cancellation is combined with small values.
Rewrite three-mass coefficients for $\mathcal{M}^1_H(4P)$

- Rewrite $s_{3K_1}^{−}$ using identities of $\gamma_±$:

$$s_{3K_1}^{−} = -\frac{\gamma_- s_{34} m_H^2}{\gamma_+ (\gamma_- - m_H^2 s_{12}) (m_H^2 + \gamma_+)} \left( \frac{m_H^2 s_{12} s_{34}}{\gamma_- + s_{12}} - (s_{14} + s_{24} + s_{34}) \gamma_+ \right)$$

- $s_{3K_1}^{−}$ is explicitly proportional to the $s_{34}$ and there are no large cancellations

- $s_{4K_1}^{−}$ in $C_{3;1234|12|34}(H, 1^+_g, 2^-_g, 3^-_g, 4^-_g)$ has similar issue

- $C_{3;1234|12|34}(H, 1^-_q, 2^+_q, 3^-_g, 4^-_g)$ and $C_{3;1234|41|23}(H, 1^-_q, 2^+_q, 3^-_g, 4^-_g)$
Rewrite cut-completion terms for $\mathcal{M}^1_H(4P)$

- Counting the order of divergence ($\Delta^{-1}$) in single soft limit ($p_2 \to 0 \sim \Delta^2$):

  $$\langle 2i \rangle, [2i] \sim \Delta, \quad s_{i2} \sim \Delta^2$$

- The overall divergence of the $\mathcal{M}^1_H(4P)$ amplitude should be $\mathcal{O}(\Delta^{-2})$, however there are terms of $\mathcal{O}(\Delta^{-4})$ inside $\mathcal{M}^1_H(4P)$:

  $$C_4(1^+_g, 2^+_g, 3^+_g, 4^-_g) \sim$$
  $$+ \frac{\langle 34 \rangle \langle 41 \rangle (3s_{124} \langle 34 \rangle \langle 41 \rangle + \langle 24 \rangle \langle 3 | p_H | 1 \rangle \langle 42 \rangle)}{3 \langle 42 \rangle^2} \hat{L}_2 (s_{124}, s_{12})$$
  $$+ \left( \frac{2s_{124} \langle 34 \rangle^2 \langle 41 \rangle^2}{\langle 24 \rangle^3} - \frac{\langle 24 \rangle \langle 3 | p_H | 1 \rangle^2}{3s_{124} \langle 42 \rangle} \right) \hat{L}_1 (s_{124}, s_{12})$$
  $$+ \frac{\langle 3 | p_H | 1 \rangle (4s_{124} \langle 34 \rangle \langle 41 \rangle + \langle 3 | p_H | 1 \rangle (2s_{14} + s_{24}))}{s_{124} \langle 24 \rangle \langle 42 \rangle^3} \hat{L}_0 (s_{124}, s_{12})$$

  $$R_4(1^+_g, 2^+_g, 3^+_g, 4^-_g) \sim \frac{[14]^2 \langle 43 \rangle^2}{2s_{12} \langle 42 \rangle^2}$$
Rewrite cut-completion terms for $\mathcal{M}_H^1(4P)$

The cut-completion terms are defined as

$$
\hat{L}_3(s, t) = L_3(s, t) - \frac{1}{2(s-t)^2} \left( \frac{1}{s} + \frac{1}{t} \right),
$$

$$
\hat{L}_2(s, t) = L_2(s, t) - \frac{1}{2(s-t)} \left( \frac{1}{s} + \frac{1}{t} \right),
$$

$$
\hat{L}_1(s, t) = L_1(s, t), \quad \hat{L}_0(s, t) = L_0(s, t), \quad L_k(s, t) = \frac{\log(s/t)}{(s-t)^k}
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\]

Identities for cut-completion terms

\[
s\hat{L}_3(s, t) = t\hat{L}_3(s, t) + \hat{L}_2(s, t), \\
s\hat{L}_2(s, t) = t\hat{L}_2(s, t) + \hat{L}_1(s, t) - \frac{1}{2} \left( \frac{1}{s} + \frac{1}{t} \right), \\
\frac{1}{s} \hat{L}_1(s, t) = \hat{L}_2(s, t) - \frac{t}{s} \hat{L}_2(s, t) + \frac{1}{2s} \left( \frac{1}{s} + \frac{1}{t} \right), \\
s\hat{L}_1(s, t) = t\hat{L}_1(s, t) + \hat{L}_0(s, t), \quad \frac{1}{s} \hat{L}_0(s, t) = \hat{L}_1(s, t) - \frac{t}{s} \hat{L}_1(s, t)
\]
Rewrite cut-completion terms for $\mathcal{M}_H^1(4P)$

Inserting $\langle 3|p_H|1 \rangle = -\langle 32 \rangle[31] - \langle 34 \rangle[41]$ into $C_4(1^+_g, 2^-, 3^-, 4^-)$:
Rewrite cut-completion terms for $\mathcal{M}_H^1(4P)$

- Inserting $\langle 3|p_H|1 \rangle = -\langle 32|31 \rangle - \langle 34|41 \rangle$ into $C_4(1^+_g, 2^-_g, 3^-_g, 4^-_g)$:

$$C_4(1^+_g, 2^-_g, 3^-_g, 4^-_g) \sim \frac{\langle 34 \rangle^2[41]^2}{3[42]^2} \left( + 3s_{124} \hat{L}_2 - \frac{3}{s_{124}} \hat{L}_0 ight. \\
\left. + \frac{1}{s_{24}} (6s_{124} \hat{L}_1 - 6\hat{L}_0 - \frac{6s_{12}}{s_{124}} \hat{L}_0) \right) \sim \mathcal{O}(\Delta^{-4})$$

- Repeat using identities for cut-completion terms, we can rewrite:

$$\frac{\langle 34 \rangle^2[41]^2}{3[42]^2} \left( 3s_{124} \hat{L}_2 - \frac{3}{s_{124}} \hat{L}_0 \right) =$$

$$\frac{\langle 34 \rangle^2[41]^2}{[42]^2} \left( s_{12} \hat{L}_2 + \frac{s_{12}}{s_{124}} \hat{L}_1 - \frac{1}{2} \left( \frac{1}{s_{124}} + \frac{1}{s_{12}} \right) \right) \sim \mathcal{O}(\Delta^{-2}) - \frac{\langle 34 \rangle^2[41]^2}{2s_{12}[42]^2}$$

$$\frac{\langle 34 \rangle^2[41]^2}{3[42]^2 s_{24}} \left( 6s_{124} \hat{L}_1 - 6\hat{L}_0 - \frac{6s_{12}}{s_{124}} \hat{L}_0 \right) = \frac{\langle 34 \rangle^2[41]^2}{[42]^2 s_{24}} \left( \frac{2s_{12}^2}{s_{124}} \hat{L}_1 \right) \sim \mathcal{O}(\Delta^0)$$
Rewrite cut-completion terms for $\mathcal{M}_H^1(4P)$

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C_4(1^+_g, 2^-, 3^-, 4^-) \sim \frac{\langle 34 \rangle^2[41]^2}{3[42]^2} \left( 3s_{124} \hat{L}_2 - \frac{3}{s_{124}} \hat{L}_0 
+ \frac{1}{s_{24}} (6s_{124} \hat{L}_1 - 6\hat{L}_0 - \frac{6s_{12}}{s_{124}} \hat{L}_0) \right) \sim O(\Delta^{-4})
\]

Repeat using identities for cut-completion terms, we can rewrite:

\[
\frac{\langle 34 \rangle^2[41]^2}{3[42]^2} \left( 3s_{124} \hat{L}_2 - \frac{3}{s_{124}} \hat{L}_0 \right) = \\
\frac{\langle 34 \rangle^2[41]^2}{[42]^2} \left( s_{12} \hat{L}_2 + \frac{s_{12}}{s_{124}} \hat{L}_1 - \frac{1}{2} \left( \frac{1}{s_{124}} + \frac{1}{s_{12}} \right) \right) \sim O(\Delta^{-2}) - \frac{\langle 34 \rangle^2[41]^2}{2s_{12}[42]^2}
\]

\[
\frac{\langle 34 \rangle^2[41]^2}{3[42]^2s_{24}} \left( 6s_{124} \hat{L}_1 - 6\hat{L}_0 - \frac{6s_{12}}{s_{124}} \hat{L}_0 \right) = \frac{\langle 34 \rangle^2[41]^2}{[42]^2s_{24}} \left( \frac{2s_{12}^2}{s_{124}} \hat{L}_1 \right) \sim O(\Delta^0)
\]

Cancellation with $R_4(1^+_g, 2^-, 3^-, 4^-) \sim \frac{[14]^2\langle 43 \rangle^2}{2s_{12}[42]^2}$
Improve numerical stability of matrix elements

- After rewriting of $C_4(1_g^+, 2_g^-, 3_g^-, 4_g^-)$ and $R_4(1_g^+, 2_g^-, 3_g^-, 4_g^-)$:

  \[
  \begin{align*}
  &\text{Figure: } |M^1_H(gggg)|^2 \text{ unstable} \\
  &\text{Figure: } |\hat{M}^1_H(gggg)|^2 \text{ stable}
  \end{align*}
  \]

- $|M^1_H(qgg\bar{q})|^2$ and $|\hat{M}^1_H(gggg)|^2$ can achieve spike plots with the same treatment
Summary

- Precise QCD calculations require amplitudes for higher orders.
- More and more efforts are required to obtain amplitudes with more loops and legs. We also need these amplitudes to be IR stable in unresolved P.S.
- Subtraction terms from phenomenology studies can be used to test the IR behaviour of amplitudes.
- Examples of identifying large cancellations and rewriting amplitudes are introduced in this talk.
- Testing of more amplitudes are needed.
Back up slides
Improve numerical stability of matrix elements

Xuan Chen  (Physics Institute, UZH)  Improve Numerical Stabilities of Amplitudes in Precise QCD Calculations  July 6-10, 2015  20 / 20