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Quantum Bayesianism, tomography, and random numbers

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What is a quantum state?

$|\psi\rangle$ expresses an agent's Bayesian degrees of belief about the consequences of his actions.

$|\psi\rangle$ is a function of the agent and the world.

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Bayesian statistics

Hypotheses: H_0, H_1

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Bayesian statistics

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Prior degrees of belief: $\Pr(H_0)$, $\Pr(H_1)$

Model: $\Pr(T|H_0)$, $\Pr(T|H_1)$

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Bayesian statistics

Hypotheses: H_0, H_1

Prior degrees of belief: $\Pr(H_0)$, $\Pr(H_1)$

Model: $\Pr(T|H_0)$, $\Pr(T|H_1)$

Data: $T_{\rm obs}$

Posterior: $\Pr(H_0|T_{\rm obs}) = \dfrac{\Pr(T_{\rm obs}|H_0)\Pr(H_0)}{\Pr(T_{\rm obs})}$.

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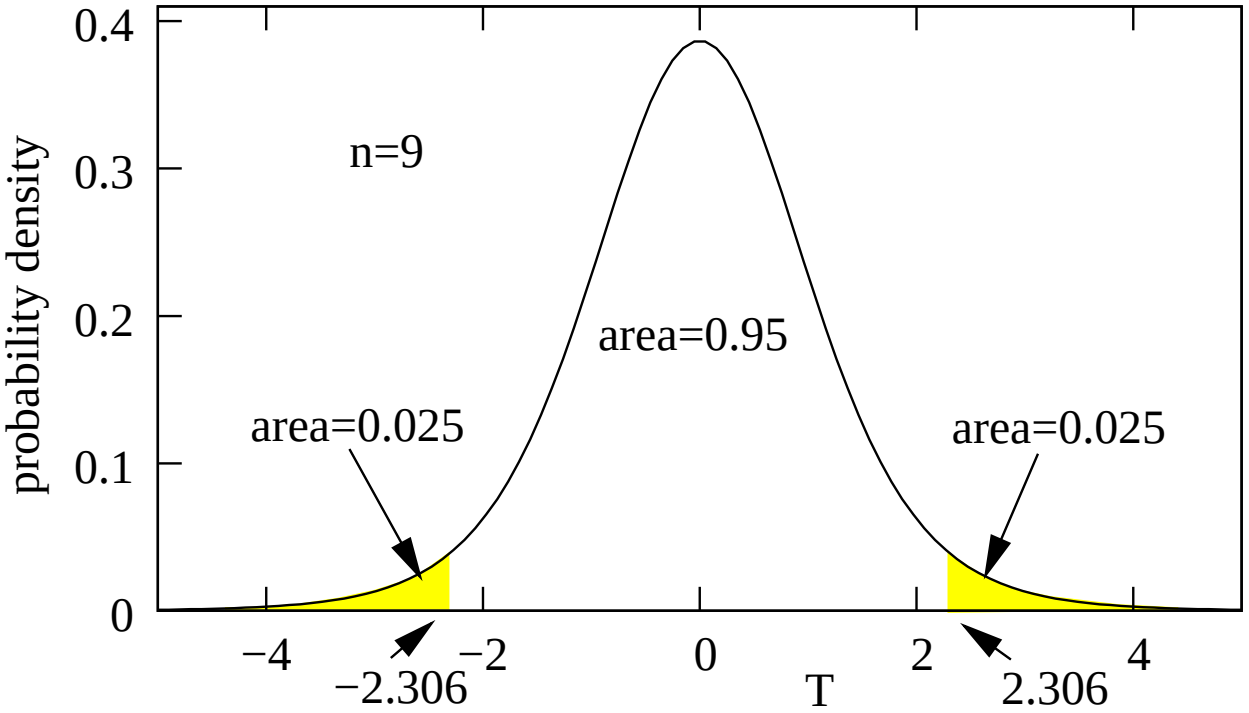
Frequentist statistics

Model: $\Pr(T|H_0)$



Find $P = \Pr(|T| > |T_{\rm obs}|)$.

Reject H_0 if $P < 5\%$

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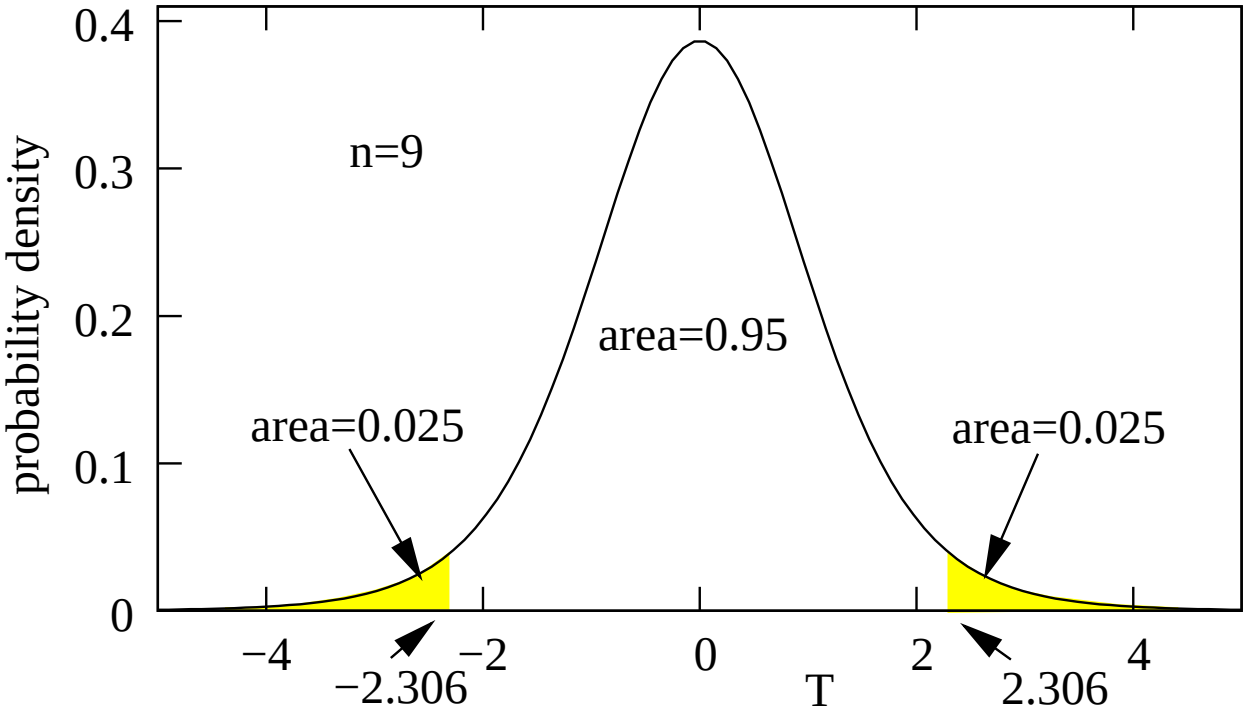
Frequentist statistics

Model: $\Pr(T|H_0)$



Find $P = \Pr(|T| > |T_{\rm obs}|)$.
Reject H_0 if $P < 5\%$

"a hypothesis that may be true may be rejected because it has not predicted observable results that have not occurred" (H. Jeffreys, 1961)

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The quantum-mechanical core

(a) Prior belief: The agent writes down his prior quantum state.

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The quantum-mechanical core

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(c) Quantum mechanics: The agent uses the quantum rules to update his quantum state.

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The quantum-mechanical core

(a) Prior belief: The agent writes down his prior quantum state.

(b) Action: The agent acts on the world to elicit a measurement outcome.

(c) Quantum mechanics: The agent uses the quantum rules to update his quantum state.

Quantum mechanics does not provide a rule for choosing the prior quantum state.

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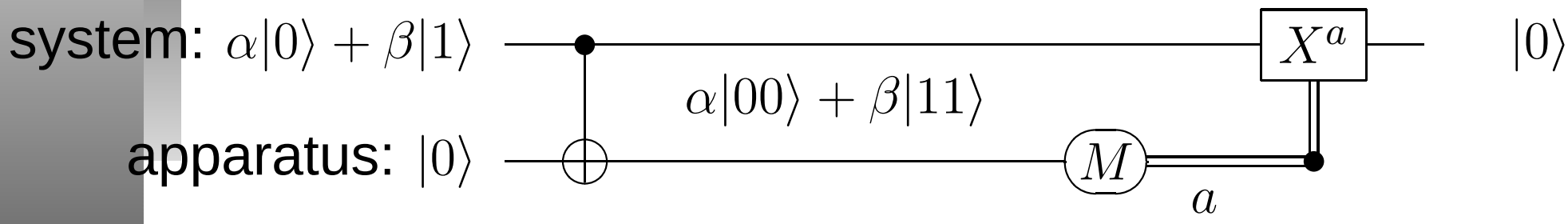
State preparation



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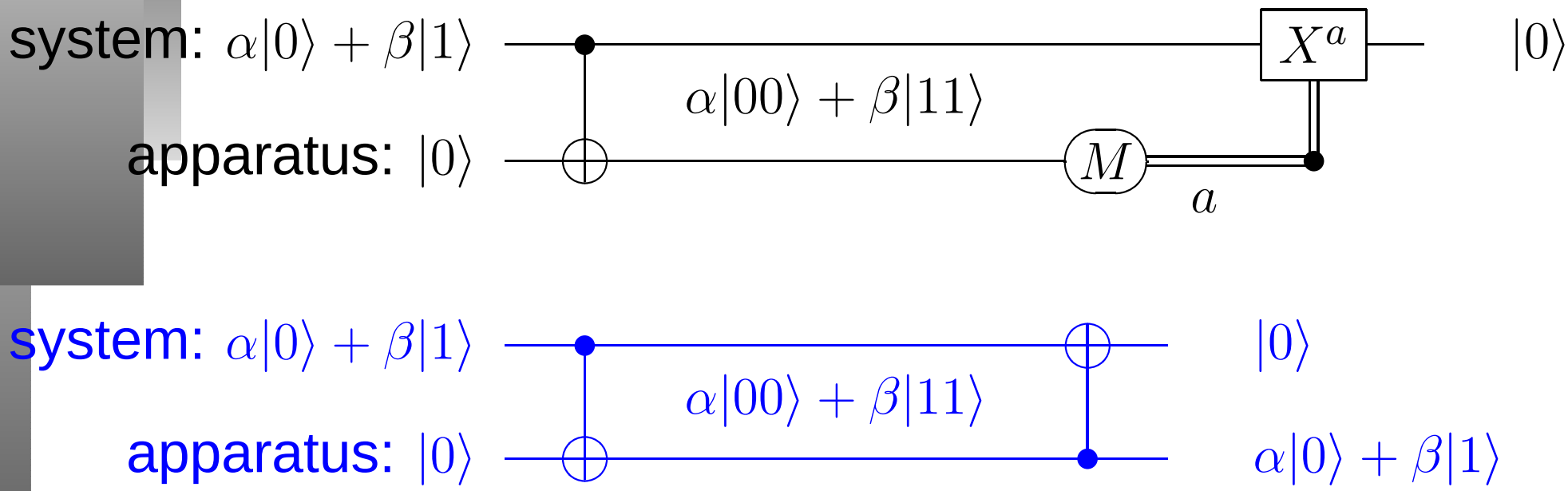
State preparation



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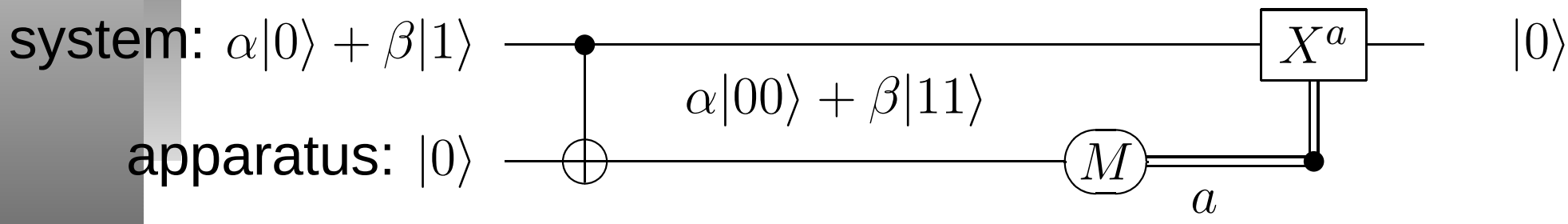

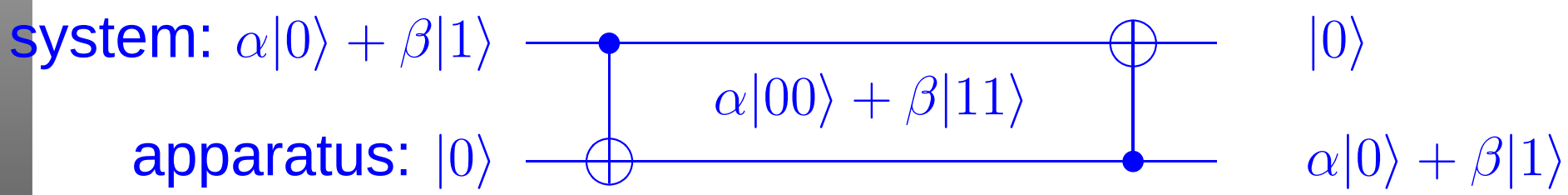

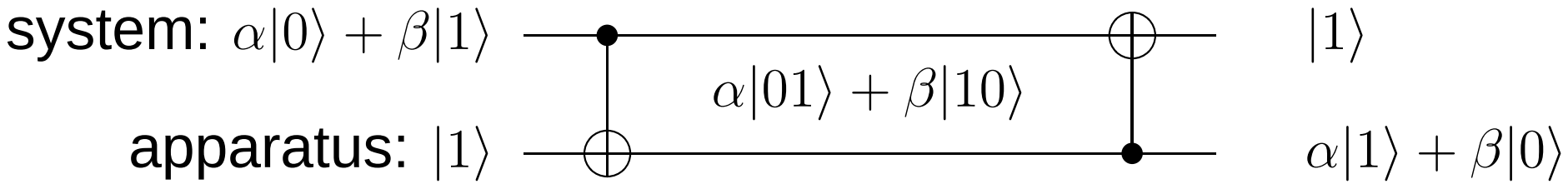

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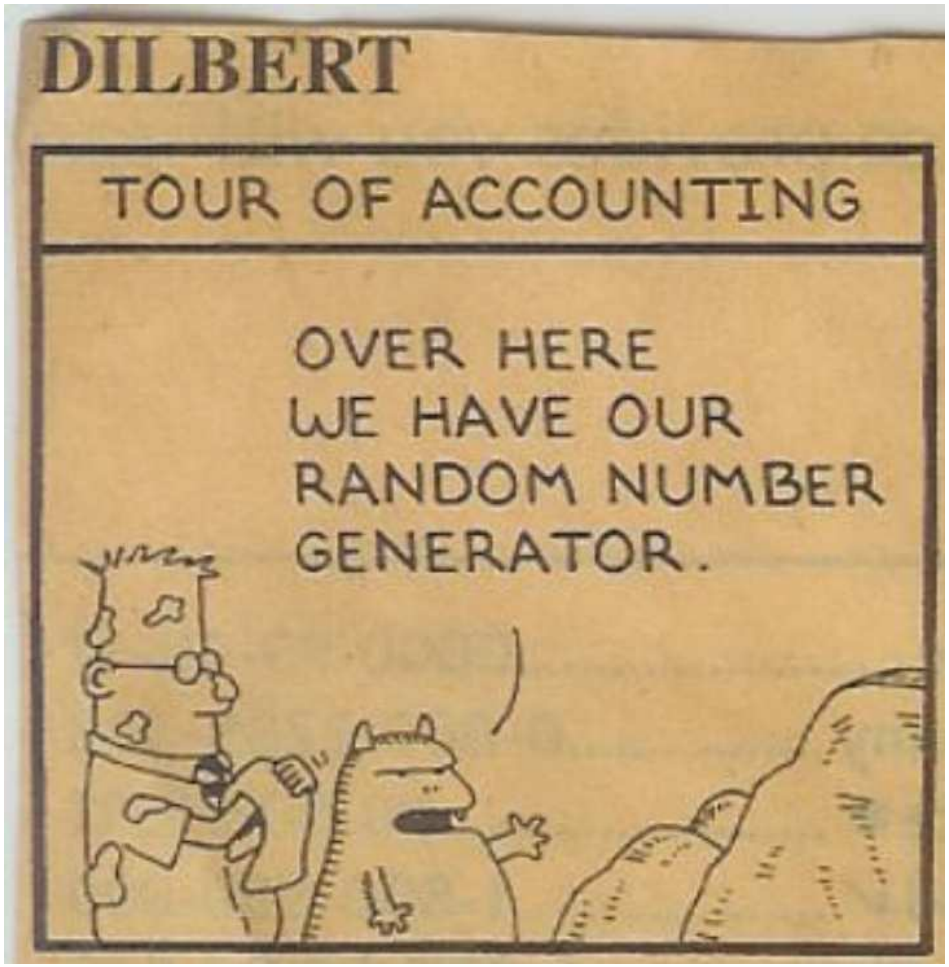

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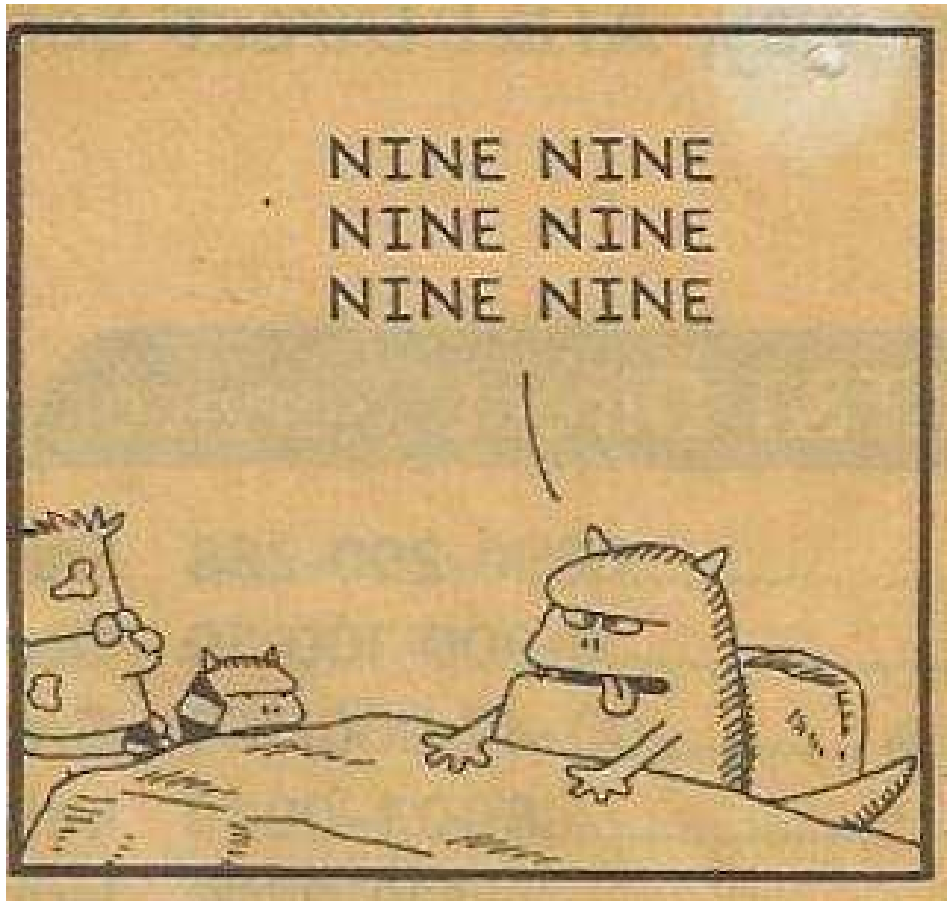

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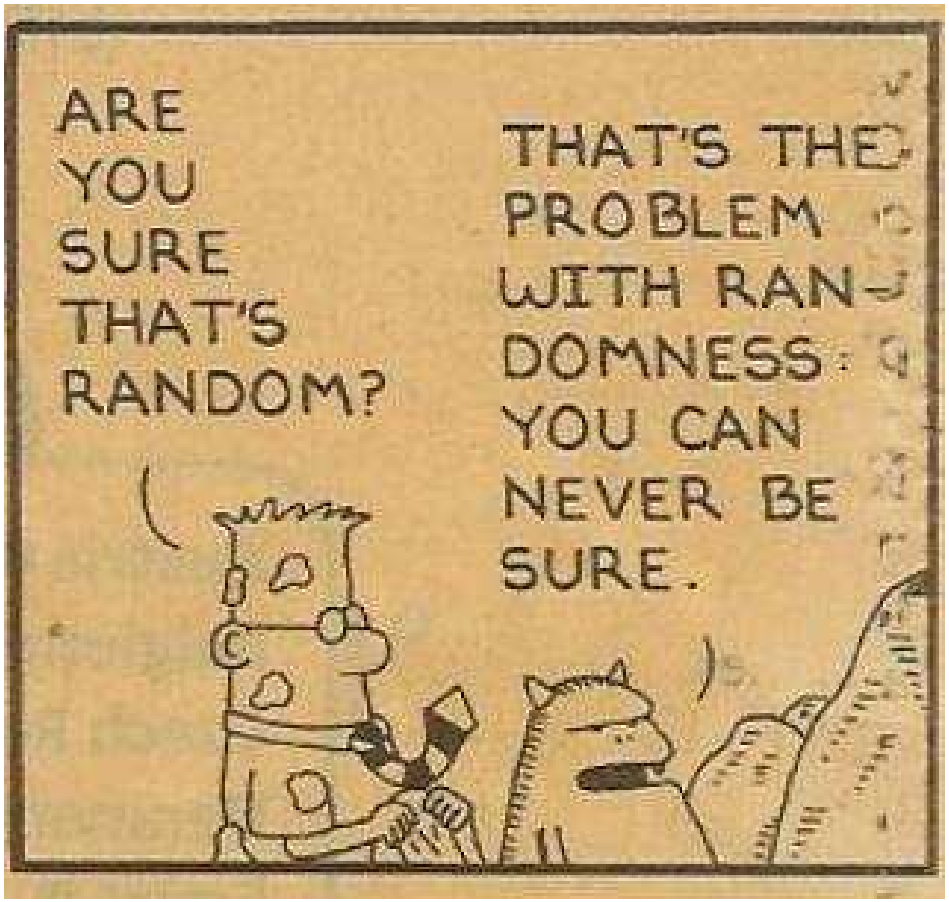

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Quantum random numbers (©www.randomnumbers.info)

0 1 1 0 1 0 0 1 1 0 0 1 1 0 0 1 1 1 0 0 1 1 0 1 1
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0 1 1 1 1 1 0 1 1 0 0 1 0 0 0 1 1 0 1 0 1 1 1 0 1
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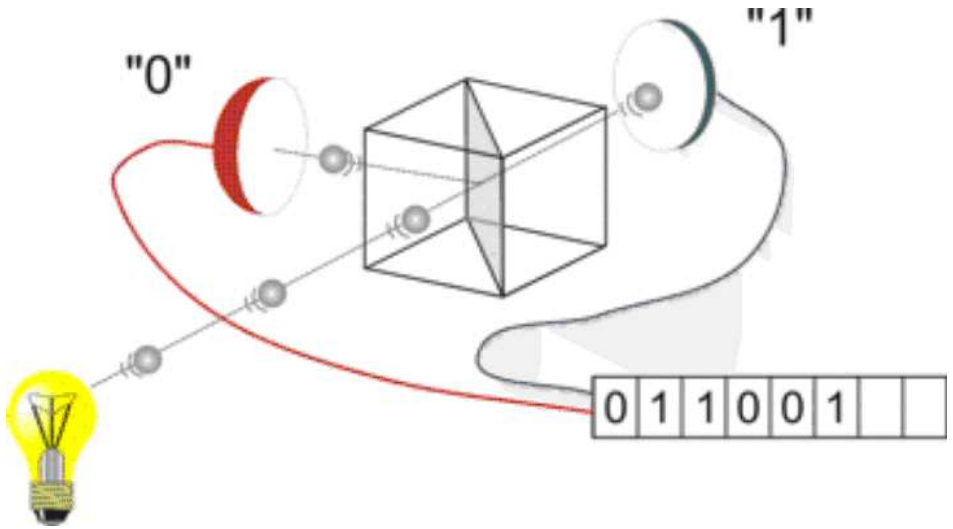
Quantum random numbers



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Quantum random numbers

$|\psi\rangle = \frac{1}{\sqrt{2^n}}(|0\rangle + |1\rangle)^{\otimes n}$ (“Prior”)

Measurement $\rightarrow x \in \{0,1\}^n$ (“Action”)

$\Pr(x) = 2^{-n}$

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Bayesian quantum tomography

$$\rho^{(N+M)} = \int d\rho \; p(\rho) \; \rho^{\otimes(N+M)}$$

measure N subsystems

get outcome $\vec{\alpha}$

$$\rho^{(M)} = \int d\rho \; p(\rho|\vec{\alpha}) \; \rho^{\otimes M}$$

$p(\rho|\vec{\alpha})$ given by a quantum Bayes rule.

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Quantum Bayes rule: detail

$$p(\rho|\vec{\alpha}) = \frac{p(\rho)p(\vec{\alpha}|\rho)}{p_\alpha} \ ,$$

where $p(\vec{\alpha}|\rho) = \mathrm{tr}\big(\, \rho^{\otimes N}\, E_{\alpha_1} \otimes \cdots \otimes E_{\alpha_N}\big)$,

$\vec{\alpha} = (\alpha_1, \ldots, \alpha_N)$,

$\{E_k\}$ is a POVM,

and $p_\alpha = \int d\rho\; p(\rho)\, p(\vec{\alpha}|\rho)$.

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Bayesian quantum tomography: practicalities

$p(\rho|\vec{\alpha})$ is a distribution on a D^2-dimensional space.

Don't compute (or store) it:

Instead, use MCMC (Metropolis-Hastings algorithm) to obtain quantities of interest directly from $\vec{\alpha}$ and $p(\rho)$ (the prior).

(See Robin Blume-Kohout, quant-ph, 2006)

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Frequentist quantum tomography

Given POVM $\{E_k\}$ and outcome $\vec{\alpha} = (\alpha_1, \ldots, \alpha_N)$, find $\rho = \rho_{ML}$ such that

$p(\vec{\alpha}|\rho) = \mathrm{tr}\big(\, \rho^{\otimes N}\, E_{\alpha_1} \otimes \cdots \otimes E_{\alpha_N}\big)$ is maximal.

(This is the maximum-likelihood estimator.)

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Three priors: 1

Exchangeable (de Finetti) prior:

$$\rho^{(N)} = \int d\rho \; p(\rho) \; \rho^{\otimes N}$$

HHHHHHHHHHHHHHHHHHHHHHHHHHHHHH

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Three priors: 2

Rosenkrantz and Guildenstern prior:

$$\rho^{(N)} = 1^{\otimes N}$$

HHHHHHHHHHHHHHHHHHHHHHHHHHHHHH?

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Three priors: 3

Counter-inductive prior:

$$\rho^{(N)} = c_0 \sum_{k=1}^{N} 2^{-k^2} \Big(|0\rangle\langle 0|^{\otimes k} \otimes |1\rangle\langle 1|^{\otimes (N-k)} + |1\rangle\langle 1|^{\otimes k} \otimes |0\rangle\langle 0|^{\otimes (N-k)} \Big)$$

HHHHHHHHHHHHHHHHHHHHHHHHHHHHHHHHHT

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Three priors: 3

State of the remaining $N-m$ qubits after getting m times 0 in a $\{|0\rangle, |1\rangle\}$ measurement:

$$\rho^{(N-m)} = c_m 2^{-m^2} |1\rangle\langle 1|^{\otimes(N-m)} + \sum_{k=1}^{N-m} c_m 2^{-(m+k)^2} |0\rangle\langle 0|^{\otimes k} \otimes |1\rangle\langle 1|^{\otimes(N-m-k)}$$

where $c_m 2^{-m^2}$ approaches 1 as m increases.

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True randomness?

n bits from quantum device: $\Pr(x) = 2^{-n}$

n bits from tossing a coin: $\Pr(x) = 2^{-n}$

What is the difference?

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Two players

One trial,

one sample space, S,

two agents, A and B,

two beliefs $P_A : S \to [0,1]$ and $P_B : S \to [0,1]$.

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Insider states

P_B is an insider state with respect to P_A if

(i) there exists an event E such that $P_B(E) = 1$ and $P_A(E) < 1$, and

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Insider states

P_B is an insider state with respect to P_A if

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There exists an insider state w.r.t. any nontrivial P_A.

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Quantum mechanics

Given: A single system $\mathcal{S}$.

2 agents $\alpha = A, B$

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Quantum mechanics

Given: A single system $\mathcal{S}$.

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2 state assignments ρ_A, ρ_B

"Agent α assigns the state ρ_α to $\mathcal{S}$."

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Quantum mechanics

Given: A single system $\mathcal{S}$.

2 agents $\alpha = A, B$

2 state assignments ρ_A, ρ_B

"Agent α assigns the state ρ_α to $\mathcal{S}$."

POVM $\{E_k\}$

${p_\alpha}^k = \mathrm{tr}(E_k \rho_\alpha)$
probability assigned by agent α to the outcome k

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Insider states

ρ_B is an insider state w.r.t. ρ_A

if

(i) There exists a POVM $\{E_k\}$ and an outcome k such that ${p_B}^k = 1$ and ${p_A}^k < 1$, and

(ii) There exists no POVM $\{E_k\}$ and outcome k such that ${p_A}^k = 1$ and ${p_B}^k < 1$.

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Quantum random numbers

There exists no insider state w.r.t. a pure state $|\psi\rangle$.

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Quantum random numbers

There exists no insider state w.r.t. a pure state $|\psi\rangle$.

Nobody has insider information about my quantum random numbers.

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Rule (i)

(i) $P \geq 0$

Worth \$1 if E is true

ticket price $\$q < 0$

A is willing to sell the ticket for a negative amount of money. Sure loss!

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Rule (ii)

(ii) $P(E) = 1$ if A believes that E is certain to occur.

Worth \$1 if E is true

ticket price $\$q < \$1$

A is willing to sell a ticket—which is definitely worth $\$1$ to her—for less than $\$1$. Sure loss!

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Rule (iii)

(iii) $P(E \vee F) = P(E) + P(F)$ if A believes that E and F are mutually exclusive.

Let $H = E \vee F$, $E \wedge F = \emptyset$.

Worth \$1 if H is true	ticket price \$q
Worth \$1 if E is true	ticket price \$r
Worth \$1 if F is true	ticket price \$s

E.g., A would buy the blue ticket for $\$q$ and sell the green tickets for $\$r + \$s$. If $q > r + s$, sure loss!.

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Rule (iv)

(iv) $P(E \wedge F) = P(E|F)P(F)$

Worth \$1 if $E \wedge F$
Worth $\$P(E|F)$ if $\neg F$

price $\$P(E|F)$

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Rule (iv)

(iv) $P(E \wedge F) = P(E|F)P(F)$

Worth \$1 if $E \wedge F$ Worth $\$P(E\|F)$ if $\neg F$	price $\$P(E\|F)$
Worth \$1 if $E \wedge F$	price $\$P(E \wedge F)$

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Rule (iv)

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Worth \$1 if $E \wedge F$ Worth $\$P(E\|F)$ if $\neg F$	price $\$P(E\|F)$
Worth \$1 if $E \wedge F$	price $\$P(E \wedge F)$
Worth $\$P(E\|F)$ if $\neg F$	price $\$P(E\|F)P(\neg F)$

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Rule (iv)

(iv) $P(E \wedge F) = P(E|F)P(F)$

Worth \$1 if $E \wedge F$ Worth $\$P(E\|F)$ if $\neg F$	price $\$P(E\|F)$
Worth \$1 if $E \wedge F$	price $\$P(E \wedge F)$
Worth $\$P(E\|F)$ if $\neg F$	price $\$P(E\|F)P(\neg F)$

Consistency implies

$\$P(E|F) = \$P(E \wedge F) + P(E|F)P(\neg F)$

Rule (iv) follows using $P(\neg F) = 1 - P(F)$.