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PROBABILISTIC THEORIES: WHAT IS SPECIAL ABOUT QUANTUM MECHANICS?

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Abstract. Quantum Mechanics (QM) is a very special probabilistic theory, yet we don't know which operational principles make it so. Here I will analyze the possibility of deriving QM as the mathematical representation of a *fair operational framework*, i. e. a set of rules which allows the experimenter to make predictions on future *events* on the basis of suitable *tests*, e. g. without interferences from uncontrollable sources. Two postulates need to be satisfied by any fair operational framework: NSF: *no-signaling from the future* (for the possibility of making predictions on the basis of past tests) FAITH: *existence of faithful states* (for the possibility of calibrating all tests and of preparing any state). I will show that all theories satisfying NSF admit a C*-algebra representation of events as linear transformations of effects. Based on a very general notion of dynamical independence, it is easy to see that all such probabilistic theories are *no-signaling without interaction* (shortly *no-signaling*)—another requirement for a fair operational framework. Postulate FAITH then implies the *local observability principle*, along with the tensor-product structure for the linear spaces of states and effects.

What is special in QM is that also *effects make a C**-*algebra*. More precisely, this is true for all hybrid quantum-classical theories, corresponding to QM plus super-selection rules. However, whereas the sum of effects can be operationally defined, the notion of effect abhors any kind of composition. Based on another very natural postulate—AE: atomicity of evolution—along with a purely mathematical postulate—CJ: Choi-Jamiolkowski isomorphism—it is possible to identify effects with atomic events, through which we can then define composition. In this way only the quantum-classical hybrid is selected within the large arena of probabilistic theories, which includes for example the so-called PR-boxes.

The presence of a purely mathematical postulate in the present QM axiomatization is a weak point in common with all previous axiomatization attempts in the literature. It is hoped, however, that the CJ postulate will be easier to derive from operational principles.

1. Introduction

After more than a century from its birth, Quantum Mechanics (QM) remains mysterious. We still don't have general principles from which to derive its remarkable mathematical framework, as it happened for the amazing Lorentz's transformations, lately re-derived by Einstein's from invariance of physical laws on inertial frames, and constancy of the light speed. But if the mathematical construction of QM doesn't follow from a general principle, then why not a more general framework?

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The deepest of the earlier attacks on the problem were those of von Neumann and coworkers [BvN36, JvNW34], who attempted to derive QM from a set of axioms with as much physical significance as possible. The general idea in Ref. [BvN36] is to derive QM as a new kind of *prepositional calculus*, a proposal that spawned the research line of *quantum logic* originated by Mackey [Mac63] (see also [Var62] and Piron [Pir76]), who gave the propositional-calculus the mathematical form of an orthomodular attice, postulating such mathematical structure as a specific *ad hoc* axiom. The work [JvNW34] considered the possibility of a commutative algebra of observables in which one needs only to define squares and sums of observables—the so-called *Jordan product* of observables a and b: $a \circ b := (a+b)^2 - a^2 - b^2$. However, such product is generally non associative and non distributive with respect to the sum, and the quantum formalism follows only with additional axioms with no clear physical significance, e. g. a distributivity axiom for the Jordan product. Segal [Seg47] later constructed a (almost) fully operational framework (assuming the sum of observables) which allows generally non-distributive algebras of observables. The consequent mathematical framework, however, is largely more general than QM. As a result of his investigation, the purely algebraic formulation of QM gathered popularity, versus the original Hilbert-space axiomatization.

In the algebraic axiomatization of QM, a physical system is defined by its C^*-algebra of observables (with identity), and the states of the system are identified with normalized positive linear functionals over the algebra, corresponding to the probability rules of measurements of observables. Indeed, the C^*-algebra of observables is more general than QM, and includes Classical Mechanics as a special case. It describes any quantum-classical hybrid, equivalent to QM with super-selection rules. Since operationally two observables are not distinguishable if they always exhibit the same probability distributions, w. l. g. one can always take the set of states as *full*—it is also said that *states separate observables*—in the sense that there are no different states having the same probability distribution for all observables. Conversely *observables separate states*, i. e. there are no different observables having the same probability distribution for all states. Notice that, in principle, there exist different observables with the same expectation for all states, but higher moments will be different.[1]

The algebra of observables is generally considered to be more "operational" than the usual Hilbert-space axiomatization, however, little is gained more than a representation-independent mathematical framework. Indeed, the algebraic framework is unable to provide operational rules for how to measure sums and products of non-commuting observables.[2] The sum of two observables cannot be given an operational meaning, since a procedure involving the measurements of the two addenda would unavoidably assume that their respective measurements are jointly executable on the same system—i. e. the observables are *compatible*. The same reasoning holds for the product of two observables. A sum-observable defined as the one giving the sum of expectations for all states [Str05] is clearly non unique,

[1]This is not the case when one considers only *sharp observables*, for which there always exists a state such that the expectation of any function of the observable equals the function of the expectation. However, we cannot rely on such concept to define the general notion of observable, since we cannot reasonably assume its feasibility (actual measurements are non sharp).

[2]The spectrum of the sum is certainly not the sum of the original spectra, e. g. the spectra of xp_y and yp_x are both $\mathbb{R}$, whereas the angular momentum component $xp_y - yp_x$ has discrete spectrum. The same is true for the product.

due to the mentioned existence of observables having the same expectation for all states, but with different higher moments (see also footnote 1). The only well defined procedures are those involving single observables, such as the measurement of a *function of a single observable*, which operationally consists in simply taking the function of the outcome.

The Jordan symmetric product has been regarded as a great advance in view of an operational axiomatization, since, in addition to being Hermitian (observables are Hermitian), is defined only in terms of squares and sums of observables—i. e. without products. There is, however, still the sum of observables in the definition of $a \circ b$. But, remarkably, Alfsen and Shultz [AS01, AS02] succeeded in deriving the Jordan product from solely geometrical properties of the convex set of states—e. g. orientability and faces shaped as Euclidean balls—however, still with no operational meaning. The problem with the Jordan product is that, in addition to being non necessarily associative, it is also not even distributive, as the reader can easily check himself. It turns out that, modulo a few topological assumptions, the Jordan algebras can be embedded in $\mathsf{Lin}(\mathsf{H})$, whereby $a \circ b = ab + ba$, however, such assumptions are still non operational. For further critical overview of these earlier attempts to an operational axiomatization of QM, the reader is also addressed to the recent books of Strocchi [Str05] and Thirring [Thi04].

In the last years the new field of Quantum Information has renewed the interest on the problem of deriving QM from operational principles. In his seminal paper [Har01] Hardy derived QM from five "reasonable axioms", which, more than being truly operational, are motivated on the basis of simplicity and continuity criteria, with the assumption of a finite number of perfectly discriminable states. In particular, his axiom 4 is purely mathematical, and is directly related to the tensor product rule for composite systems. In another popular paper [CBH03], Clifton Bub and Halvorson have shown how three fundamental information-theoretic constraints—(a) the no-signaling, (b) the no-broadcasting, (c) the impossibility of unconditionally secure bit commitment—suffice to entail that the observables and state space of a physical theory are quantum-mechanical. Unfortunately, the authors already started from a C^*-algebraic framework for observables, which, as already discussed, has little operational basis, and already coincides with the quantum-classical hybrid. Therefore, more than deriving QM, their informational principles just force the algebra of any individual system to be non-Abelian.

In Ref. [D'A07c][3] I showed how it is possible to derive the formulation of QM in terms of observables represented as Hermitian operators over Hilbert spaces with the right dimensions for the tensor product, starting from few operational axioms. However, it is not clear jet if such framework is sufficient to identify QM (or the quantum-classical hybrid) as the only probability theory resulting from axioms. Later, in Refs. [D'A07b, D'A07a, D'A07d] I have shown how a C^*-algebraic framework for transformations (not for observables!) naturally follows from an operational probabilistic framework.

A very recent and promising point of view for attacking the problem of QM axiomatization consists in positioning QM within the landscape of general probabilistic theories, including theories with nonlocal correlations stronger than the quantum ones, such as those of the so-called *PR-boxes* boxes [PR94]. Such theories, although

[3]Most of the results of Ref. [D'A07c] were originally conjectured in Refs. [D'A06c] and [D'A06a].

no-signaling, violate the quantum Tsirelson bound [Cir80]. Within the framework of the PR-boxes general versions of the no-cloning and no-broadcasting theorems have been proven [BBLW07]. In Ref. [Bar05] it has been shown that certain features, usually thought of as specifically quantum, are present in all except classical theories [Bar05]. These include the non-unique decomposition of a mixed state into pure states, disturbance on measurement (related to the possibility of secure key distribution), and no-cloning, whereas more recently necessary and sufficient conditions have been established for teleportation [BBLW08]. In all these works Quantum Information has inspired task-oriented axioms to be considered in a general operational framework that can incorporate QM, classical theory, and other no-signaling probabilistic theories (for an illustration of this general point of view see also Ref. [Bar06]).

In the present paper I will consider the possibility of deriving QM as the mathematical representation of a *fair operational framework*, i. e. a set of rules which allows the experimenter to make predictions on future *events* on the basis of suitable *tests*, in a spirit close to Ludwig's axiomatization [Lud85]. *States* are simply the compendia of probabilities for all possible outcomes of any test. I will consider a very general class of probabilistic theories, and examine the consequences of two Postulates that need to be satisfied by any fair operational framework:

NSF: *no-signaling from the future*, implying that it is possible to make predictions based on present tests;

FAITH: *existence of faithful states*, implying that it is possible to calibrate any test and to prepare any state.

NSF is implicit in the definition itself of conditional probabilities for cascade-tests, entailing that *events are identified with transformations*, whence *evolution is identified with conditioning*. As we will see, the *effects* of Ludwig are the equivalence classes of events occurring with the same probability for all states. For them I will show how we can introduce operationally a linear-space structure. I will then show how all theories satisfying NSF admit a C*-algebra representation of events as linear transformations of effects.

Based on a very general notion of dynamical independence, entailing the definition of local state, it is immediate to see that all these theories are *no-signaling*, which is the current way of saying that the theories satisfy the principle of *Einstein locality*, namely that there can be no detectable effect on a system from whatever is done on another non interacting system. This is clearly another requirement for a fair operational framework. The postulate FAITH then implies the *local observability principle*, along with the tensor-product structure for the linear spaces of states and effects, plus some other nice mathematical consequences, such as the isomorphism of cones of states and effects, a weaker version of the quantum self-duality. The local observability principle means that we can always perform an informationally complete test using only local tests—another requirement for a fair operational framework.

But, what is special about QM is that not only transformations, but also *effects make a C*-algebra*. More precisely, this is true for all hybrid quantum-classical theories, i. e. corresponding to QM plus super-selection rules. However, whereas the sum of effects can be operationally defined, their composition has no operational meaning, since the notion itself of "effect" abhors any kind of composition. I will then show that by another very natural postulate:

AE: atomicity of evolution;

along with the mathematical postulate

CJ: Choi-Jamiolkowski isomorphism [Cho75, Jam72];

it is possible to identify effects with "atomic" events, (i. e. elementary events) through which we can then define their composition. In this way one selects the quantum-classical hybrid among all possible general probabilistic theories (e. g. the PR-boxes, which satisfy both NSF and FAITH).

The presence of a purely mathematical postulate—the CJ's—is a limitation common to all previous attempts of operational axiomatization of QM. However, due to its relative simplicity, it is hoped that the CJ postulate will be easier to derive from pure operational principles.

2. C*-algebra representation of single-system probabilistic theories

2.1. **Tests and states.** In a probabilistic operational framework a **system** SYS is identified with a collection $\mathsf{SYS} = \{\mathbb{A}, \mathbb{B}, \mathbb{C}, \ldots\}$ of possible **tests** [4] $\mathbb{A}, \mathbb{B}, \mathbb{C}, \ldots$ each one being a set $\mathbb{A} = \{\mathscr{A}_i\}$, $\mathbb{B} = \{\mathscr{B}_j\}$, $\mathbb{C} = \{\mathscr{C}_k\}$, ... of **events** $\mathscr{A}_i, \mathscr{B}_j, \mathscr{C}_k, \ldots$ occurring probabilistically. SYS also includes all tests that are random choices of different tests, whence it is a convex set. The **state** ω describing the preparation of the system is the probability rule $\omega(\mathscr{A})$ for all possible events $\mathscr{A} \in \mathbb{A}$ occurring in any possible test $\mathbb{A} \in \mathsf{SYS}$.[5] For each test $\mathbb{A}$ we have the completeness $\sum_{\mathscr{A}_j \in \mathbb{A}} \omega(\mathscr{A}_j) = 1$, in particular $\omega(\mathscr{D}) = 1$ for any deterministic event $\mathscr{D}$, such as the free evolution. We will denote by $\mathfrak{S}(\mathsf{SYS})$ (shortly $\mathfrak{S}$) the set of all possible states of the system, including all states that are random choices of different states, whence $\mathfrak{S}$ is a convex set. We will also write shortly "for all possible states ω" meaning $\forall \omega \in \mathfrak{S}(\mathsf{SYS})$, and we will do similarly for other operational objects.

2.2. **Conditioning and transformations.** The **cascade** $\mathbb{B} \circ \mathbb{A}$ of two tests $\mathbb{A} = \{\mathscr{A}_i\}$ and $\mathbb{B} = \{\mathscr{B}_j\}$ performed on the same system is the new test with events $\mathbb{B} \circ \mathbb{A} = \{\mathscr{B}_j \circ \mathscr{A}_i\}$, where $\mathscr{B} \circ \mathscr{A}$ denotes the **composite event** made of the event $\mathscr{A}$ followed by the event $\mathscr{B}$. The notation $\omega(\mathscr{B} \circ \mathscr{A})$ means the joint probability of the composite event $\mathscr{B} \circ \mathscr{A}$. We now give our first Postulate:

[4] The present notion of test corresponds to that of **experiment** of Ref. [D'A07c]. Quoted from the same reference: "An experiment on an object system consists in making it interact with an apparatus, which will produce one of a set of possible events, each one occurring with some probability. The probabilistic setting is dictated by the need of experimenting with partial *a priori* knowledge about the system (and the apparatus). In the logic of performing experiments to predict results of forthcoming experiments in similar preparations, the information gathered in an experiment will concern whatever kind of information is needed to make predictions, and this, by definition is the *state* of the object system at the beginning of the experiment. Such information is gained from the knowledge of which transformation occurred, which is the "outcome" signaled by the apparatus."

[5] By definition the state is the knowledge of the variables of a system sufficient to make predictions. In the present context, it allows to predict the results of tests, whence it is the probability rule for all events in any conceivable test.

Postulate NSF (No signaling from the future). *The marginal probability of any event $\mathscr{A}$ is independent from any forthcoming test $\mathbb{B}$ in cascade, and is equal to the probability for no forthcoming event, namely*

$$\sum_{\mathscr{B}_j \in \mathbb{B}} \omega(\mathscr{B}_j \circ \mathscr{A}) =: f(\mathbb{B}, \mathscr{A}) \equiv \omega(\mathscr{A}), \quad \forall \mathbb{B} \in \mathsf{SYS}, \forall \omega \in \mathfrak{S}(\mathsf{SYS}). \tag{1}$$

The NSF Postulate[6] will guarantee the normalization of the conditioned probability $p(\mathscr{B}|\mathscr{A}) = \omega(\mathscr{B} \circ \mathscr{A})/\omega(\mathscr{A})$ of occurrence of event $\mathscr{B}$ conditional on the previous occurrence of event $\mathscr{A}$. The conditioning sets a new probability rule corresponding to the notion of **conditional state** $\omega_{\mathscr{A}}$, which gives the probability that an event occurs knowing that event $\mathscr{A}$ has occurred with the system prepared in the state ω, namely $\omega_{\mathscr{A}} \doteq \omega(\cdot \circ \mathscr{A})/\omega(\mathscr{A})$[7]. We can now regard the event $\mathscr{A}$ as transforming with probability $\omega(\mathscr{A})$ the state ω to the (unnormalized) state $\mathscr{A}\omega = \omega(\cdot \circ \mathscr{A})$.[8] Therefore, Postulate NSF entails the identifications:

event≡transformation $\Longrightarrow$ evolution≡state-conditioning.[9]

Notice that a transformation $\mathscr{A}$ is completely specified by the probability rule $\omega(\cdot \circ \mathscr{A})$ for all states ω. In particular the **identity transformation** $\mathscr{I}$ is completely specified by the rule $\omega(\cdot \circ \mathscr{I}) = \omega \ \forall \omega \in \mathfrak{S}$.

In the following we will denote the set of all possible transformations/events by $\mathfrak{T}(\mathsf{SYS})$, shortly $\mathfrak{T}$. The convex structure of SYS entails a convex structure for $\mathfrak{T}$, whereas the cascade of tests entails the composition of transformations. The latter, along with the existence of the identity transformation $\mathscr{I}$, gives to $\mathfrak{T}$ the structure of *convex monoid*.

2.3. **Effects.** From the notion of conditional state two complementary types of equivalences for transformations follow: the *dynamical* and the *informational* equivalence. The transformations $\mathscr{A}_1$ and $\mathscr{A}_2$ are **dynamically equivalent** when $\omega_{\mathscr{A}_1} = \omega_{\mathscr{A}_2} \ \forall \omega \in \mathfrak{S}$, namely when they produce the same conditional state for all prior states ω. On the other hand, the transformations $\mathscr{A}_1$ and $\mathscr{A}_2$ are **informationally equivalent** when $\omega(\mathscr{A}_1) = \omega(\mathscr{A}_2) \ \forall \omega \in \mathfrak{S}$, namely when they occur with the same probability. Operationally, a transformation $\mathscr{A}$ is completely specified by all the joint probabilities in which it is involved, whence, it is univocally given by the probability rule $\omega(\cdot \circ \mathscr{A})$. It follows that two transformations will be operationally

[6]In principle the marginal probability in Eq. (1) depends on the choice of the second test $\mathbb{B}$ of the cascade. This has been noticed for the first time by Masanao Ozawa (private communication). Postulate NSF is needed to properly define conditional probabilities for cascade tests, from which the notion of conditional state follows. The latter, in turn, is intimately related to the notion of "effect" and to the action of "transformations" over effects (see the following). Of course one can assume the opposite postulate of no-signaling from the past, considering conditioning from the future instead from the past, thus reversing the arrow of time.

[7]Throughout the paper the central dot "$\cdot$" denotes the location of the pertinent variable.

[8]This is the same as the notion of *quantum operation* in QM, which gives the conditioning $\omega_{\mathscr{A}} = \mathscr{A}\omega/\mathscr{A}\omega(\mathscr{I})$, or, in other words, the analogous of the quantum Schrödinger picture evolution of states.

[9]Clearly the identification *evolution≡state-conditioning* also includes the deterministic case $\mathscr{D}\omega = \omega(\cdot \circ \mathscr{D})$ of transformations $\mathscr{D}$ with $\omega(\mathscr{D}) = 1 \, \forall \omega \in \mathfrak{S}$—the analogous of quantum channels, including unitary evolutions.

indistinguishable—whence they will be identified (denoted as $\mathscr{A}_1 = \mathscr{A}_2$)—when they are both dynamically and informationally equivalent.[10]

The informational equivalence class is the **effect**[11]. In the following we will denote effects with lowercase letters $a, b, c, \ldots$ and use the underlined symbol $\underline{\mathscr{A}}$, or also $[\mathscr{A}]_{\mathrm{eff}}$, to denote the effect containing transformation $\mathscr{A}$. We will also write $\mathscr{A}_0 \in \underline{\mathscr{A}}$ meaning that "the transformation $\mathscr{A}_0$ belongs to the equivalence class $\underline{\mathscr{A}}$", or "$\mathscr{A}_0$ has effect $\underline{\mathscr{A}}$", or "$\mathscr{A}_0$ is informationally equivalent to $\mathscr{A}$". The **deterministic effect** will be denoted by e, corresponding to $\omega(e) = 1$ for all states ω. We will denote the set of effects for a system SYS as $\mathfrak{E}(\mathsf{SYS})$, shortly $\mathfrak{E}$. Notice that, by definition, effects are positive linear functionals over the set of states $\mathfrak{S}$ bounded by 1. Therefore, by duality, we have a convex structure for $\mathfrak{E}$.[12] Notice that since operationally two effects must be identified when they have the same probability for all possible states, and, conversely, any two states must be identified if they produce the same probability for all possible effects, namely **the set of states $\mathfrak{S}$ is separating for effects $\mathfrak{E}$**, and, viceversa, **the set of effects $\mathfrak{E}$ is separating for states $\mathfrak{S}$**.

By definition $\omega(\mathscr{A}) \equiv \omega(\underline{\mathscr{A}})$, whence we legitimately write $\omega(\underline{\mathscr{A}})$ instead of $\omega(\mathscr{A})$. The identity $\omega_{\mathscr{A}}(\mathscr{B}) \equiv \omega_{\mathscr{A}}(\underline{\mathscr{B}})$ implies that $\omega(\mathscr{B} \circ \mathscr{A}) = \omega(\underline{\mathscr{B}} \circ \mathscr{A})$ for all states ω, leading to the chaining rule $\underline{\mathscr{B}} \circ \mathscr{A} \in \underline{\mathscr{B} \circ \mathscr{A}}$ corresponding to the "Heisenberg picture" evolution in terms of transformations acting on effects. Notice that transformations act on effects from the right, inheriting the composition rule of transformations ($\mathscr{B} \circ \mathscr{A}$ means "$\mathscr{A}$ followed by $\mathscr{B}$"). Consistently, in the "Schrődinger picture", we have $\mathscr{B}\omega(\cdot \circ \mathscr{A}) = \omega(\cdot \circ \mathscr{B} \circ \mathscr{A})$, corresponding to $(\mathscr{B} \circ \mathscr{A})\omega = \omega(\cdot \circ \mathscr{B} \circ \mathscr{A})$.

2.4. **Observables.** An **observable** $\mathbb{L}$ is a complete set of effects $\mathbb{L} = \{l_i\}$ summing to the deterministic effect as $\sum_{l_i \in \mathbb{L}} l_i = e$, namely l_i are the effects of the events of a test. An observable $\mathbb{L} = \{l_i\}$ is named **informationally complete** for SYS when each effect can be written as a real linear combination of l_i, namely $\mathsf{Span}_{\mathbb{R}}(\mathbb{L}) = \mathsf{Span}_{\mathbb{R}}[\mathfrak{E}(\mathsf{SYS})]$. When the effects of $\mathbb{L}$ are linearly independent the informationally complete observable is named *minimal*. Clearly, since $\mathfrak{E}$ is separating for states, **any informationally complete observable separates states**, namely using an informationally complete observable we can reconstruct also any state $\omega \in \mathfrak{S}(\mathsf{SYS})$ from the set of probabilities $\omega(l_i)$. The existence of a minimal informationally complete observable constructed from the set of available tests is guaranteed by the following Theorem:

Theorem 1 (Existence of minimal informationally complete observable)**.** *It is always possible to construct a minimal informationally complete observable for* SYS *out of a set of tests of* SYS.

For the proof see Ref. [D'A07b].

2.5. **Linear structures for transformations and of effects.** Transformations $\mathscr{A}_1$ and $\mathscr{A}_2$, for which one has the bound $\omega(\mathscr{A}_1) + \omega(\mathscr{A}_2) \leqslant 1$, $\forall \omega \in \mathfrak{S}$ can in principle occur in the same test, and we will call them **test-compatible**. For test-compatible transformations one can define their addition $\mathscr{A}_1 + \mathscr{A}_2$ via the probability

[10] This immediately follows from the identity $\omega(\mathscr{B} \circ \mathscr{A}_1) = \omega(\mathscr{B} \circ \mathscr{A}_2)$, $\forall \omega \in \mathfrak{S}$, $\forall \mathscr{B} \in \mathfrak{T}$.

[11] This is the same notion of "effect" introduced by Ludwig [Lud85]

[12] In a different way we can say that effects inherit the convex structure from transformations.

rule

$$\omega(\cdot\circ(\mathscr{A}_1+\mathscr{A}_2))=\omega(\cdot\circ\mathscr{A}_1)+\omega(\cdot\circ\mathscr{A}_2), \tag{2}$$

corresponding to the union-event. The corresponding informational and dynamical classes are then given by

$$\forall\omega\in\mathfrak{S}:\ \omega(\mathscr{A}_1+\mathscr{A}_2)=\omega(\mathscr{A}_1)+\omega(\mathscr{A}_2),\quad (\mathscr{A}_1+\mathscr{A}_2)\omega=\mathscr{A}_1\omega+\mathscr{A}_2\omega. \tag{3}$$

Notice distributivity of composition "$\circ$" w.r.t. addition "$+$". We can also define the multiplication $\lambda\mathscr{A}$ of a transformation $\mathscr{A}$ by a scalar $0\leqslant\lambda\leqslant 1$ by the rule

$$\omega(\cdot\circ\lambda\mathscr{A})=\lambda\omega(\cdot\circ\mathscr{A}), \tag{4}$$

namely $\lambda\mathscr{A}$ is the transformation dynamically equivalent to $\mathscr{A}$, but occurring with rescaled probability $\omega(\lambda\mathscr{A})=\lambda\omega(\mathscr{A})$. It follows that for every couple of transformation $\mathscr{A}$ and $\mathscr{B}$ the transformations $\lambda\mathscr{A}$ and $(1-\lambda)\mathscr{B}$ are test-compatible for $0\leqslant\lambda\leqslant 1$, consistently with the convex structure already introduced for the set of transformations $\mathfrak{T}$. The notions of *i)* test-compatibility, *ii)* sum and *iii)* multiplication by a scalar, are then inherited from transformations to effects via informational equivalence.

For mathematical convenience we can extend the convex monoid of transformations $\mathfrak{T}$ to its cone $\mathfrak{T}_+$ and to the linear real and complex spans $\mathfrak{T}_\mathbb{R}=\mathsf{Span}_\mathbb{R}(\mathfrak{T})$ and $\mathfrak{T}_\mathbb{C}=\mathsf{Span}_\mathbb{C}(\mathfrak{T})$, respectively, via the *Cartesian decomposition* $\mathfrak{T}_\mathbb{C}=\mathfrak{T}_\mathbb{R}\oplus i\mathfrak{T}_\mathbb{R}$ (i. e. each element $\mathscr{A}\in\mathfrak{T}_\mathbb{C}$ can be uniquely written as $\mathscr{A}=\mathscr{A}_R+i\mathscr{A}_I$, with $\mathscr{A}_R,\mathscr{A}_I\in\mathfrak{T}_\mathbb{R}$). Likewise for states and effects, we define in an analogous way $\mathfrak{S}_+,\mathfrak{S}_\mathbb{R},\mathfrak{S}_\mathbb{C}$ and $\mathfrak{E}_+,\mathfrak{E}_\mathbb{R},\mathfrak{E}_\mathbb{C}$, respectively. We will also simply call "effects" also the generalized effects that are elements of the cone $\mathfrak{E}_+$, and call **real (complex) effects** the elements of $\mathfrak{E}_\mathbb{R}$ ($\mathfrak{E}_\mathbb{C}$), and likewise for states. Due to the mentioned state-effect duality we have the linear-space identifications $\mathfrak{S}_\mathbb{R}\equiv\mathfrak{E}_\mathbb{R}$ and $\mathfrak{S}_\mathbb{C}\equiv\mathfrak{E}_\mathbb{C}$. Note that the cones of states and effects are **pointed**, i. e. they include the origin (null vector of the linear space). Moreover, one has $\omega=0$ iff $\omega(e)=0$ (otherwise $\lambda\omega$ is a state for some $\lambda\geqslant 0$), whereas an effect $a=0$ iff $\omega(a)=0$ for all states, whence iff $\vartheta(e)$=0 for a **chaotic state** ϑ, i. e. a state that can be written as the convex combination of any state with some other state.

On the real linear spaces we can introduce the respective **natural norms** $\|a\|:=\sup_{\omega\in\mathfrak{S}}|\omega(a)|$, $\|\omega\|:=\sup_{a\in\mathfrak{E}_\mathbb{R},\|a\|\leqslant 1}|\omega(a)|$, $\|\mathscr{A}\|:=\sup_{a\in\mathfrak{E}_\mathbb{R},\|a\|\leqslant 1}\|\underline{\mathscr{B}}\circ\mathscr{A}\|$. Closures in norm (again for mathematical convenience) make $\mathfrak{E}_\mathbb{R}$ and $\mathfrak{S}_\mathbb{R}$ a dual Banach pair, and $\mathfrak{T}_\mathbb{R}$ a real Banach algebra.[13] Therefore, all operational quantities can be mathematically represented as elements of such Banach spaces.

2.6. **The C* algebra of transformations.** It is easy to represent the transformations as elements of $\mathfrak{T}_\mathbb{C}$ regarded as a complex C* algebra. This is trivial, since $\mathfrak{T}_\mathbb{C}$ are by definition linear transformations of effects, making an associative sub-algebra $\mathfrak{T}_\mathbb{C}\subseteq\mathsf{Lin}(\mathfrak{E}_\mathbb{C})$ of the matrix algebra over $\mathfrak{E}_\mathbb{C}$. **Adjoint** and **norm** can be easily defined in terms of any chosen real **scalar product** $(\,,)$ **over** $\mathfrak{E}_\mathbb{C}$, with the adjoint defined as $(a\circ\mathscr{A}^\dagger,b)=(a,b\circ\mathscr{A})$, and the norm as $\|\mathscr{A}\|=\sup_{a\in\mathfrak{E}_\mathbb{C}}\|a\circ\mathscr{A}\|$, with $\|a\|=\sqrt{(a,a)}$. (Notice that these norms are different from the "natural norms" previously defined). In such way $\mathfrak{T}_\mathbb{C}$ is automatically a C*-algebra. Indeed, upon

[13] An algebra of maps over a Banach space inherits the norm induced by that of the Banach space on which it acts. It is then easy to prove that the closure of the algebra under such norm is a Banach algebra.

reconstructing $\mathfrak{E}_\mathbb{C}$ and $\mathfrak{T}_\mathbb{C}$ from the original real spaces via the Cartesian decomposition $\mathfrak{E}_\mathbb{C} = \mathfrak{E}_\mathbb{R} \oplus i\mathfrak{E}_\mathbb{R}$ and $\mathfrak{T}_\mathbb{C} = \mathfrak{T}_\mathbb{R} \oplus i\mathfrak{T}_\mathbb{R}$, and introducing the scalar product on $\mathfrak{E}_\mathbb{C}$ as the sesquilinear extension of a real symmetric scalar product $(,)_\mathbb{R}$ over $\mathfrak{E}_\mathbb{R}$, the adjoint of a real element $\mathscr{A} \in \mathfrak{T}_\mathbb{R}$ is just the transposed matrix $\mathscr{A}^t$ with respect to a real basis orthonormal for $(,)_\mathbb{R}$, and $\mathscr{A}^\dagger := \mathscr{A}_R{}^t - i\mathscr{A}_I{}^t$ for a general $\mathscr{A} = \mathscr{A}_R + i\mathscr{A}_I \in \mathfrak{T}_\mathbb{C}$. A natural choice of matrix representation for $\mathfrak{T}_\mathbb{R}$ is given by its action over a minimal informational complete observable $\mathbb{L} = \{l_i\}$ (the scalar product $(,)_\mathbb{R} := (,)_\mathbb{L}$ will correspond to declare $\mathbb{L}$ as orthonormal). Upon expanding $[l_i \circ \mathscr{A}]_{\text{eff}}$ again over $\mathbb{L} = \{l_i\}$ one has the matrix representation $l_i \circ \mathscr{A} = \sum_j \mathscr{A}_{ji} l_j$. Using the fact that $\mathbb{L}$ is state-separating, we can write the probability rule as the pairing $\omega(a) = (\omega, a)_\mathbb{R}$ between $\mathfrak{E}_\mathbb{R}$ and $\mathfrak{S}_\mathbb{R}$ (and analogously for their complex spans).[14]

In this way we have seen that for every probabilistic theory we can always represent transformations/events as elements of the C*-algebra $\mathfrak{T}_\mathbb{C}$ made of matrices acting on the linear space of complex effects $\mathfrak{E}_\mathbb{C}$. In Fig. 1 the logical derivation of the C*-algebra representation of the theory is summarized.

Conversely, given a C*-algebra $\mathfrak{T}_\mathbb{C}$ with the cone of transformations $\mathfrak{T}_+$, along with the components of the vector e representing the deterministic effect (e. g. an informationally complete observable), we can rebuild the probabilistic theory by constructing the cone of effects as the orbit[15] $\mathfrak{E}_+ = e \circ \mathfrak{T}_+$, and taking the cone of states $\mathfrak{S}_+$ as the dual cone of $\mathfrak{E}_+$.

3. Probabilistic theories for many independent systems

3.1. **Dynamical independence and local states.** A purely dynamical notion of *system independence* coincides with the possibility of performing local tests. Precisely, we will call systems SYS_1 and SYS_2 **independent** if there exists *local tests*, say $\mathbb{A}^{(1)} \in \mathsf{SYS}_1$ and $\mathbb{A}^{(2)} \in \mathsf{SYS}_2$, whose transformations commute each other, namely [16]

$$\mathscr{A}^{(1)} \circ \mathscr{B}^{(2)} = \mathscr{B}^{(2)} \circ \mathscr{A}^{(1)},\ \forall \mathscr{A}^{(1)} \in \mathbb{A}^{(1)},\ \forall \mathscr{B}^{(2)} \in \mathbb{B}^{(2)}. \tag{5}$$

Since the local transformations commute, we will just put them in a string, e. g. $(\mathscr{A}, \mathscr{B}, \mathscr{C}, \ldots) := \mathscr{A}^{(1)} \circ \mathscr{A}^{(2)} \circ \mathscr{A}^{(3)} \circ \ldots$ Clearly since the probability $\omega(\mathscr{A}, \mathscr{B}, \mathscr{C}, \ldots)$ is independent on the time ordering of transformations, it is just a function only of

[14] The present derivation of the C*-algebra representation of transformations is more direct than that in Ref. [D'A07b], and is just equivalent to the probabilistic framework inherent in the notion of "test" (see also the summary of the whole logical deduction in the flow chart in Fig. 1). The derivation in Ref. [D'A07b] needed the two additional postulates: a) the existence of dynamically independent systems; b) the existence of faithful symmetric bipartite states. However, the added value of that derivation is that the C*-algebra is also equipped with two involutions—an operational "transposed" and a complex conjugation for transformations—with the composition of the two giving the adjoint. Both involutions where defined in terms of the faithful symmetric state, and the notion of adjoint generally depends on it. Here the notion of adjoint depends on the choice of the informationally complete observable $\mathbb{L}$—or just a basis for $\mathfrak{E}_\mathbb{R}$—in terms of which one can define the transposed of the matrix representation $\mathscr{A}^t$ of the transformation, along with the scalar product $(b, a)_\mathbb{L}$. The relations between the present informationally-complete-based and the faithful-state-based C* representations are certainly an interesting subject, and will be analyzed in detail in a thorough future report. Here we can just say that the basis $\mathbb{L}$ would play the role of the Jordan canonical basis for the faithful state.

[15] The "orbit" $e \circ \mathfrak{T}_+$ is defined as the set $\{e \circ \mathscr{A} | \mathscr{A} \in \mathfrak{T}_+\}$.

[16] The present definition of independent systems is purely dynamical, in the sense that it does involve statistical requirements, e. g. the existence of factorized states.

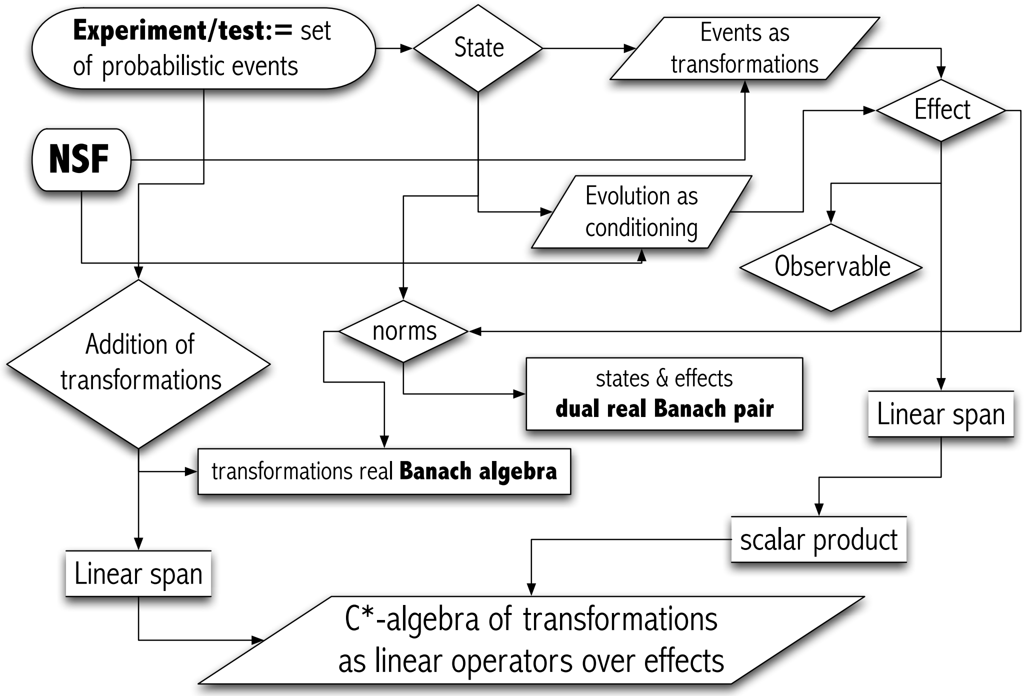

FIGURE 1. Logical flow chart leading to the representation of any probabilistic theory in terms of a C*-algebra of linear transformations over the linear space of complex effects.

the effects $\omega(\mathscr{A},\mathscr{B},\mathscr{C},\ldots) = \omega(\underline{\mathscr{A}},\underline{\mathscr{B}},\underline{\mathscr{C}},\ldots)$, namely the joint effect corresponding to local transformations is made of local effects $[(\mathscr{A},\mathscr{B},\mathscr{C},\ldots)]_{\text{eff}} \equiv (\underline{\mathscr{A}},\underline{\mathscr{B}},\underline{\mathscr{C}},\ldots)$. For N systems in the joint state Ω, we define the **local state** $\Omega|_n$ of the n-th system the probability rule for any local transformation $\mathscr{A}$ at the n-th system, with all other systems untouched, namely

$$\Omega|_n(\mathscr{A}) \doteq \Omega(\mathscr{I},\ldots,\mathscr{I},\underbrace{\mathscr{A}}_{n\text{th}},\mathscr{I},\ldots). \tag{6}$$

Clearly, since the probability for local transformations depends only on their respective effects, the local state is equivalently defined as $\Omega|_n(a) \doteq \Omega(e,\ldots,e,\underbrace{a}_{n\text{th}},e,\ldots)$, for $a \in \mathfrak{E}$. It immediately follows that the local state $\Omega|_n$ is independent on any deterministic transformation—i. e. any test—that is performed on systems different from the nth: this is exactly the general statement of the **no signaling** or **acausality of local tests**. Therefore, the present notion of dynamical independence directly implies no-signaling. [17] The definition in Eq. (6) can be trivially

[17]The present notion of dynamical independence is indeed so minimal that it can be satisfied not only by the quantum tensor product, but also by the quantum direct sum [D'A06b]. In order to extract only the tensor product an additional assumption is needed. As shown in Refs. [D'A06b, D'A07b] two possibilities are either postulating the existence of bipartite states that are

extended to joint and local unnormalized states. Then, one has the following useful lemma:

Lemma 1. *A local unnormalized state is zero iff the joint state is zero.*

Proof. Obvious, since $\Omega(e, e, \ldots, e) = \Omega_n(e) = 0$.

3.2. **Faithful states.** In the following for simplicity we will consider only identical systems, leaving the general analysis for multipartite heterogeneous systems (along with transformations from different systems) for a thorough future report. In the following we will use the notation $\mathfrak{Z}^{\times N} := \mathfrak{Z}(\mathsf{SYS}_1, \mathsf{SYS}_2, \ldots, \mathsf{SYS}_N)$ to denote N-partite sets/spaces, with $\mathfrak{Z} = \mathfrak{S}, \mathfrak{S}_+, \mathfrak{S}_{\mathbb{R}}, \mathfrak{S}_{\mathbb{C}}, \mathfrak{E}, \mathfrak{E}_+, \ldots$

A bipartite state is **dynamically faithful** (w.r.t. SYS_1) when for every local transformation $\mathscr{A}$ on SYS_1 the map $\mathscr{A} \leftrightarrow (\mathscr{A}, \mathscr{I})\Phi$ is one-to-one, namely $\forall \mathscr{A} \in \mathfrak{T}_{\mathbb{R}}$ $(\mathscr{A}, \mathscr{I})\Phi = 0 \Longleftrightarrow \mathscr{A} = 0$ (equivalently for every bipartite effect $\underline{\mathscr{B}}$ one has $\Phi(\underline{\mathscr{B}} \circ (\mathscr{A}, \mathscr{I})) = 0 \Longleftrightarrow \mathscr{A} = 0$). On the other hand, we will call a bipartite state Φ **preparationally faithful** (w.r.t. SYS_1) if every joint bipartite state Ψ can be achieved by a local transformation $\mathscr{T}_\Psi$ on SYS_1 occurring with nonzero probability, i. e. $\Psi = (\mathscr{T}_\Psi, \mathscr{I})\Phi$, with $\mathscr{T}_\Psi \in \mathfrak{T}^+$. Finally a state is simply called **faithful** (w.r.t. SYS_1), if it is both dynamically and preparationally faithful. Both kinds of faithfulness can be defined w.r.t. either $\mathsf{SYS}_{1,2}$. Dynamical and preparational faithfulness correspond to the properties of being *separating* and *cyclic* for the C*-algebra of transformations. We now have the Postulate:

Postulate FAITH (Existence of faithful states). *For any bipartite system there exist faithful states (w.r.t. either one of the two systems).*

Postulate FAITH is operationally crucial, since dynamical faithfulness is a necessary condition for the calibrability of tests, whereas preparational faithfulness is a necessary condition for preparability of states via tests. In QM every entangled state with maximal Schmidt number is faithful. However, notice that also the case of a separating set of input states e. g. for SYS_1 (as in the conventional input output process tomography) corresponds to a separable dynamically faithful state w.r.t. SYS_1.[18]

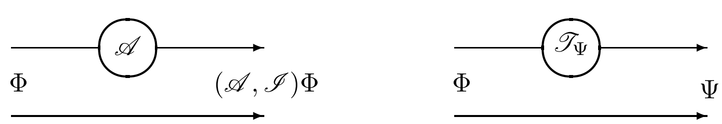

FIGURE 2. Illustration of the notions of dynamically (left figure) and preparationally (right figure) faithful state for a bipartite system.

Postulate FAITH has two immediate implications, given by the two following lemmas.

Lemma 2. *Postulate FAITH implies that the two cones $\mathfrak{E}_+$ and $\mathfrak{S}_+$ are isomorphic.*

dynamically and preparationally faithful, or postulating the local observability principle. Here we will consider the former as a postulate, and derive the latter as a theorem.

[18]Explicitly such faithful state has the form $\rho = \sum_i p_i \rho_i \otimes |i\rangle\langle i|$, $\{\rho_i\}$ denoting the separating set of states for SYS_1, $\{p_i\}$ their probabilities, and $\{|i\rangle\}$ is an orthonormal basis for SYS_2.

Proof. For Φ faithful w.r.t. SYS_1, consider the linear map $a \mapsto \omega_a := \Phi(a, \cdot)$ which to every effect $a \in \mathfrak{C}_+(\mathsf{SYS}_1)$ associates an (un-normalized) state $\omega_a \in \mathfrak{S}_+(\mathsf{SYS}_2)$. Clearly the map ω_a preserves the cone structure, since it is linear. The map is surjective, since Φ is preparationally faithful w.r.t. SYS_1, whence every joint state and in particular every local state can be obtained from a local effect. To prove injectivity, consider that the identity $\omega_a = 0$ is equivalent to $\Phi((e, \cdot) \circ (\mathscr{A}, \mathscr{I})) = 0$ for any $\mathscr{A} \in a$, and, using Lemma 1, this is equivalent to $\Phi(\cdot \circ (\mathscr{A}, \mathscr{I})) = 0$ for any $\mathscr{A} \in a$, but since the state Φ is also dynamically faithful w.r.t. SYS_1, one has that $\mathscr{A} = 0$, namely $a = 0$. We have thus shown that $\omega_a = 0 \iff a = 0$, namely the map is also injective. ■

Lemma 3. *Postulate FAITH implies that bipartite effects are a cone in the tensor product $\mathfrak{C}_{\mathbb{C}}^{\otimes 2}$, and bipartite states are a cone in $\mathfrak{S}_{\mathbb{C}}^{\otimes 2}$.*

Proof. State-effect duality implies $\mathfrak{C}_{\mathbb{C}}^{\times 2} = \mathfrak{S}_{\mathbb{C}}^{\times 2}$. Postulate FAITH implies $\mathfrak{S}_{\mathbb{C}}^{\times 2} = \mathfrak{T}_{\mathbb{C}}$. Since transformations act linearly over effects one has $\mathfrak{T}_{\mathbb{C}} \subseteq \mathsf{Lin}(\mathfrak{C}_{\mathbb{C}}) = \mathfrak{C}_{\mathbb{C}}^{\otimes 2}$, whence $\mathfrak{C}_{\mathbb{C}}^{\times 2} = \mathfrak{S}_{\mathbb{C}}^{\times 2} = \mathfrak{T}_{\mathbb{C}} \subseteq \mathfrak{C}_{\mathbb{C}}^{\otimes 2}$, and, by duality $\mathfrak{S}_{\mathbb{C}}^{\times 2} \subseteq \mathfrak{S}_{\mathbb{C}}^{\otimes 2}$. ■

Therefore, the existence of a faithful state for bipartite systems guarantees that we can represent bipartite quantities (states, effects, transformations) as elements of tensor product of the single-system spaces. This fact implies the following relevant principle

Corollary 1 (Local observability principle)**.** *For every composite system there exist informationally complete observables made of local informationally complete observables.*

Proof. A joint observable $\mathbb{J}$ made of local observables $\mathbb{L} = \{l_i\}$ on SYS_1 and $\mathbb{M} = \{m_j\}$ on SYS_2 is of the form $\mathbb{J} = \mathbb{L} \times \mathbb{M} = \{(l_i, m_j)\}$. Then, by definition, the statement of the theorem is $\mathfrak{C}_{\mathbb{C}}^{\times 2} \subseteq \mathsf{Span}_{\mathbb{C}}(\mathbb{J}) = \mathsf{Span}_{\mathbb{C}}(\mathbb{L} \times \mathbb{M}) = \mathfrak{C}^{\otimes 2}$ ■

Operationally, the Local Observability Principle plays a crucial role, since it reduces enormously the experimental complexity, by guaranteeing that only local (although jointly executed) tests are sufficient to retrieve a complete information of a composite system, including all correlations between the components. The principle reconciles holism with reductionism in a non-local theory, in the sense that we can observe a holistic nature in a reductionistic way, namely locally.

4. What is special about Quantum Mechanics as a probabilistic theory?

The mathematical representation of the operational probabilistic framework derived up to now is completely general for any fair operational framework that allows local tests, test-calibration, and state preparation. These include not only QM and classical-quantum hybrid, but also other no-signaling non-local probabilistic theories such as the *PR-boxes* theories [PR94]. What is then special about QM? The peculiarity of QM among probabilistic operational theories is that

Effects not only can be linearly combined, but also they can be composed each other, so that complex effect make a C*-algebra.

Operationally the last assertion is odd, since *the notion of effect abhors composition!* Therefore, the composition of effects (i. e. the fact that they make a C*-algebra, i. e. an operator algebra over complex Hilbert spaces) must be derived from additional postulates. What I will show here is that

With a single mathematical postulate and assuming the natural postulate of atomicity of evolution (a kind of determinism in the evolution) one can derive the composition of effects in terms of composition of atomic events.

We will then be left with the problem of translating the remaining mathematical postulate into an operational one.

Let me first clarify what I mean for *atomicity of evolution* in the operational probabilistic framework. As we have seen in Subsect. 2.5 for test-compatible events/transformations $\mathscr{A}_1$, $\mathscr{A}_2$ we can define their sum $\mathscr{A}_1+\mathscr{A}_2$ as the *coarse grained event* given by the union of the two events. We will call an event/transformation *elementary* or *atomic* if it cannot be obtained as (non trivial) sum of other events. Mathematically, **atomic events** belong to extremal rays of the cone of transformations $\mathfrak{T}_+$, and their set will be denoted by $\mathsf{Erays}(\mathfrak{T}_+)$.

We now formulate the postulate about atomicity of evolution:

Postulate AE (Atomicity of evolution). *The composition of atomic transformations is atomic.*

The above postulate is so natural that it looks obviously true. Indeed, when joining events $\mathscr{A}$ and $\mathscr{B}$ into the event $\mathscr{A}\wedge\mathscr{B}$, the latter is atomic if both $\mathscr{A}$ and $\mathscr{B}$ are atomic. However, it cannot be proved that if each of the two events $\mathscr{A}$ and $\mathscr{B}$ is atomic in every test, then the event $\mathscr{B}\circ\mathscr{A}$ is also atomic when it occurs in a single test—not in a cascade.

We now state our mathematical Postulate:

Mathematical Postulate CJ (Choi-Jamiolkowski isomorphism). *The cone of transformations is isomorphic to the cone of positive bilinear forms over complex effects* [Cho75, Jam72].

By cone isomorphism e. g. between cones $\mathfrak{C}_1$ and $\mathfrak{C}_2$ we just mean a one-to-one linear mapping between the cones, which, as a consequence, will send extremal rays of $\mathfrak{C}_1$ to extremal rays of $\mathfrak{C}_2$, and positive linear combinations to positive linear combinations. In terms of a sesquilinear scalar product over complex effects, positive bilinear forms can be regarded as a positive matrices over complex effects, i. e. elements of the cone $\mathsf{Lin}_+(\mathfrak{E}_\mathbb{C})$. The extremal rays $\mathsf{Erays}(\mathsf{Lin}_+(\mathfrak{E}_\mathbb{C}))$ are rank-one positive operators $|a\rangle\langle a|\in\mathsf{Erays}(\mathsf{Lin}_+(\mathfrak{E}_\mathbb{C}))$ with $a\in\mathfrak{E}_\mathbb{C}$. Via the surjective map $a\mapsto|a\rangle\langle a|$ from $\mathfrak{E}_\mathbb{C}$ to $\mathsf{Erays}(\mathsf{Lin}_+(\mathfrak{E}_\mathbb{C}))$ the CJ isomorphism now establishes a surjective map ς between $\mathfrak{E}_\mathbb{C}$ and $\mathsf{Erays}(\mathfrak{T}_+)$—the atomic events—as follows

$$\varsigma:\mathfrak{E}_\mathbb{C}\ni a\mapsto\mathbf{I}_{CJ}(|a\rangle\langle a|)=:\mathscr{T}_a\in\mathsf{Erays}(\mathfrak{T}_+),\tag{7}$$

$\mathbf{I}_{CJ}$ denoting any linear map realizing the Choi-Jamiolkowski isomorphism. By construction, the set of complex effects mapped to the same transformation $\mathscr{T}_a$ is given by the the set $\{e^{i\phi}a\}\subseteq\mathfrak{E}_\mathbb{C}$ of complex effects that differ only by a multiplicative phase factor $e^{i\phi}$. Therefore, the surjection ς identifies complex effects with atomic events/transformations apart from a phase. Assuming now Postulate AE, the composition of two atomic transformations is still atomic, i. e. $\mathsf{Erays}(\mathfrak{T}_+)$ is closed under composition. As long as there exists a two-cocycle $\phi(a,b)$,[19] this allows us to define composition of complex effects via the map $\mathbf{I}_{CJ}$ between cones as follows

$$\forall a,b\in\mathfrak{E}_{\mathbb{C}},\quad ba=\iota^{-1}(\mathscr{T}_b\circ\mathscr{T}_a)e^{i\phi(b,a)}, \tag{8}$$

ι denoting the surjection induced by the isomorphism $\mathbf{I}_{CJ}$.

Notice that it is always possible to choose the trivial two-cocycle identically zero. Moreover we will see soon that we can redefine the product in a way that is independent on the explicit form of the two-cocycle.

First we notice that the adjoint for complex effects is defined taking real effects $\mathfrak{E}_{\mathbb{R}}=\mathsf{Span}_{\mathbb{R}}(\mathfrak{E}_+)$ as selfadjoint, and taking the complex conjugate on Cartesian decomposition $\mathfrak{E}_{\mathbb{C}}=\mathfrak{E}_{\mathbb{R}}\oplus i\mathfrak{E}_{\mathbb{R}}$, along with the composition rule $(ba)^\dagger=a^\dagger b^\dagger$.[20]

Now that composition between complex effects is defined, all linear transformations can be written as linear combination of right and left multiplication. We now need the following simple lemma:

Lemma 4. *Given a complex linear space $\mathfrak{Z}$ and a sesquilinear involution $\mathfrak{Z}\ni a\mapsto a^*\in\mathfrak{Z}^*\simeq\mathfrak{Z}$, the tensors $a^*\otimes a\in\mathfrak{Z}^*\otimes\mathfrak{Z}$ generate the extremal rays of the cone $(\mathfrak{Z}^*\otimes\mathfrak{Z})_+:\mathsf{Co}\{a^*\otimes a,\, a\in\mathfrak{Z}\}$, whose complex span is $\mathfrak{Z}^*\otimes\mathfrak{Z}$.*

Proof. The proof uses the polarization identity

$$b^*\otimes a=\frac{1}{4}\sum_{k=0}^{3}i^k(a+i^kb)^*\otimes(a+i^kb). \tag{9}$$

■

It is an immediate consequence of Lemma 4 that the following cones in $\mathsf{Lin}_+(\mathfrak{E}_{\mathbb{C}})$ are isomorphic

$$\mathsf{Co}\{|a\rangle\langle a|,\, a\in\mathfrak{E}_{\mathbb{C}}\}\simeq\mathsf{Co}\{a^\dagger\cdot a,\, a\in\mathfrak{E}_{\mathbb{C}}\}\subseteq\mathsf{Lin}_+(\mathfrak{E}_{\mathbb{C}}), \tag{10}$$

where $a^\dagger\cdot a$ denotes right multiplication by a and left multiplication by $a^\dagger$. Therefore, according to Postulate CJ, the linear maps $a^\dagger\cdot a$, $a\in\mathfrak{E}_{\mathbb{C}}$ represent all possible atomic transformations. We now construct the algebra of complex effects starting from $\mathfrak{T}_+$. First, we fix the deterministic effect e corresponding to the identity in the associative algebra of complex effects. Then, to each effect a we associate the atomic transformation $\mathscr{T}_a=a^\dagger\cdot a$ (according to our Postulate CJ there must exists an actual atomic transformation in $\mathfrak{T}_+$ producing exactly the same linear transformation as $\mathscr{T}_a$, i. e. $x\circ\mathscr{T}_a=a^\dagger xa$, given that $\mathfrak{E}_{\mathbb{C}}$ is now an associative algebra). Now, introduce the linear transformation $\mathscr{T}_{a,b}=b^\dagger\cdot a\in\mathsf{Lin}(\mathfrak{E}_{\mathbb{C}})$, given by

$$\mathscr{T}_{a,b}:=\tfrac{1}{4}\sum_{k=0}^{3}i^k\mathscr{T}_{a+i^kb}=\mathbf{I}_{CJ}(|a\rangle\langle b|) \tag{11}$$

[19]A two-cocycle must satisfy the Jacoby associativity constraint $\phi(b,a)+\phi(c,ba)=\phi(c,b)+\phi(cb,a)$ $\forall a,b,c\in\mathfrak{E}_{\mathbb{C}}$, along with the identity $\phi(a,a^{-1})=\phi(a,\mathrm{id})=0$ for invertible a, id denoting the identity.

[20]Consistently, it is possible to choose the cocycle to obey the identity $\phi(b^\dagger,a^\dagger)=-\phi(a,b)$, giving $\phi(a^\dagger,a)=0$.

via the polar identity. We have now a one-to-one linear map between $\mathfrak{E}_\mathbb{C}$ and a linear subspace of $\mathsf{Lin}(\mathfrak{E}_\mathbb{C})$, given by

$$a = e \circ \mathscr{T}_{e,a}, \iff \mathscr{T}_{e,a} = \cdot\, a \tag{12}$$

Composition of elements of $\mathfrak{E}_\mathbb{C}$ is now given as follows

$$ab = e \circ \mathscr{T}_{e,a} \circ \mathscr{T}_{e,b}, \tag{13}$$

and the cone of effects $\mathfrak{E}_+$ is given by

$$\mathfrak{E}_+ = \{e \circ \mathscr{T}_a \equiv a^\dagger a,\ \forall a \in \mathfrak{E}_\mathbb{C}\}. \tag{14}$$

Notice that the transformations corresponding to right multiplications $\mathscr{T}_{e,a}$ form indeed an algebra with identity, however, one has $\mathscr{T}_{e,a^\dagger} = \mathscr{T}_{e,a}^\dagger$ only if the scalar product defined on $\mathfrak{E}_\mathbb{C}$ is a "trace", i. e. $(a, b) = (b^\dagger, a^\dagger)$, which is the case if it is the sesquilinear extension of a symmetric scalar product over the subspace $\mathfrak{E}_\mathbb{R}$ of selfadjoint elements. Also, notice that the full algebra $\mathsf{Lin}(\mathfrak{E}_\mathbb{C})$ is generated by the "atomic multiplicators" $\mathscr{T}_{a,b}$, which compose according to the rule $\mathscr{T}_{a,b} \circ \mathscr{T}_{a',b'} = \mathscr{T}_{aa',bb'}$. In QM, the cone isomorphism $\mathbf{I}_{CJ}$ sending rank-one elements of $\mathbf{I}_{CJ}(|a\rangle\langle b|)$ to atomic multiplicators $\mathscr{T}_{a,b}$ is explicitly given by the involution

$$\mathbf{I}_{CJ}(\mathscr{T}_{a,b})(x) = \mathrm{Tr}_1[(x \otimes I)(\mathscr{T}_{a,b} \otimes \mathscr{I})(E)E] = \mathrm{Tr}[b^\dagger x]a, \tag{15}$$

where E denotes the swap unitary operator, and $x \in \mathfrak{E}_\mathbb{C}$ is any system operator.

It is now possible to reconstruct from the probability tables of the systems the full C*-algebra of complex effects $\mathfrak{E}_\mathbb{C}$ as an operator algebra $\mathfrak{E}_\mathbb{C} \subseteq \oplus_i \mathsf{Lin}(\mathsf{H_i})$. Here is the recipe:

(1) Look for all subspaces $(\mathfrak{E}_\mathbb{R})_i$ invariant under $\mathfrak{T}_\mathbb{R}$.
For each i:
(2) build $\mathsf{Lin}(\mathsf{H_i}) \supseteq (\mathfrak{E}_\mathbb{R})_i$ with $\dim(\mathsf{H}_i) = \left\lceil \sqrt{\dim[(\mathfrak{E}_\mathbb{R})_i]} \right\rceil$, $\lceil x \rceil$ denoting the smallest integer greater than x;
(3) represent e as $\chi(e) = I$ the identity over $\oplus_i \mathsf{H}_i$;
(4) build $\mathsf{Lin}(\mathsf{Her}(\mathsf{H_i})) \supseteq (\mathfrak{T}_\mathbb{C})_i$;
(5) look for atomic transformations $\mathsf{Erays}(\mathfrak{T}_+)_i$;
(6) for each $\mathscr{A} \in \mathsf{Erays}(\mathfrak{T}_+)_i$ take an operator $A \in \mathsf{Lin}(\mathsf{H_i})$ to represent $\mathscr{A}$ as $\chi(\mathscr{A}) = A^\dagger \cdot A \in \mathsf{Lin}(\mathsf{Her}(\mathsf{H_i}))$;
(7) represent $\underline{\mathscr{A}}$ as $\chi(\underline{\mathscr{A}}) = \chi(e \circ \mathscr{A}) = A^\dagger A$;
(8) construct states as density operators using the Gleason-like theorem [Gle57] for effects [Bus03, CFMR00].

5. Conclusions

Theoretical Physics is in essence is a mathematical "representation" of reality. By "representation" we mean to explain one thing by means of another, to connect the object that we want to understand—the *thing-in-itself*—with an object that we already know well—the *standard*—comparing the two via the representation. In Theoretical Physics we lay down morphisms from structures of reality to corresponding mathematical structures: groups, algebras, vector spaces, etc, each mathematical structure capturing a different side of reality.

QM, in some way, goes differently. We have a beautiful simple mathematical structure—Hilbert spaces and operator algebras—with an unprecedented predicting

Symbol(s)	Meaning	Related quantities
$\mathsf{SYS} = \{\mathbb{A}, \mathbb{B}, \mathbb{C}, \ldots\}$	System	
$\mathbb{A}, \mathbb{B}, \mathbb{C}, \ldots$	Tests	$\mathbb{A} = \{\mathscr{A}_j\}$ Test:= set of possible events
$\mathscr{A}, \mathscr{B}, \mathscr{C}, \ldots$	Events≡transformations	
ω	states, $\mathfrak{S}$ Convex set of states	$\omega(\mathscr{A})$: probability that event $\mathscr{A}$ occurs in state ω
$\mathfrak{T}$	Convex monoid of transformations/events	$\mathfrak{T}_\mathbb{R}$, $\mathfrak{T}_\mathbb{C}$: linear spans of $\mathfrak{T}$, $\mathfrak{T}_+$: convex cone
$\underline{\mathscr{A}}$, $[\mathscr{A}]_{\rm eff}$	Effect containing event $\mathscr{A}$	
$a, b, c, \ldots$	Effects	e: deterministic effect
$\mathfrak{E}$	Convex set of effects	$\mathfrak{E}_\mathbb{R}$, $\mathfrak{E}_\mathbb{C}$: linear spans of $\mathfrak{E}$, $\mathfrak{E}_+$: convex cone
$\mathbb{L} = \{l_j\}$	observable	$\sum_{l_i \in \mathbb{L}} l_i = e$
$\mathfrak{T}_\mathbb{C}$	C*-algebra of transformations/events	
$a \circ \mathscr{T}$	Operation of transformation $\mathscr{T}$ over effect a	
$\omega_\mathscr{A}$	Conditioned states	$\omega_\mathscr{A} := \omega(\cdot \circ \mathscr{A})/\omega(\mathscr{A})$, $\mathscr{A}\omega = \omega(\cdot \circ \mathscr{A})$
$\mathsf{Lin}_+(\mathfrak{E}_\mathbb{C})$	Cone of linear maps corresponding to positive bilinear forms over $\mathfrak{E}_\mathbb{C}$	$\mathsf{Lin}_+(\mathfrak{E}_\mathbb{C}) = \{\mathscr{T} \in \mathfrak{T}_\mathbb{C} : (a, a \circ \mathscr{T}) \geqslant 0, \forall a \in \mathfrak{E}_\mathbb{C}\}$

Table 1. Summary of notation

power in the whole physical domain. However, we don't have morphisms from the operational structure of reality into a mathematical structure. In this sense, we can say that QM is still not truly a "representation" of reality.

We have analyzed the possibility of deriving QM as the mathematical representation of a *fair operational framework*, i. e. a set of rules which allows one to make predictions on future *events* on the basis of suitable *tests*. The two Postulates NSF and FAITH need to be satisfied by any fair operational framework, the former in order to be able to make predictions based on present tests, the latter to allow calibrability of any test and preparability of any state. We have seen that all theories satisfying NSF admit a C*-algebra representation of events as linear transformations of complex effects. Based on a very general notion of dynamical independence, all such theories are *no-signaling*, and the postulate FAITH implies the tensor-product structure for the linear spaces of complex states and effects. What is special about QM—and all hybrid quantum-classical theories—is that also complex effects make a C*-algebra, however, with no operational meaning for their composition. Assuming then another very natural postulate on atomicity of evolution, along with a single mathematical postulate (the Choi-Jamiolkowski isomorphism) we have been able to identify complex effects with atomic events, through which we could define their composition, obtaining the C*-algebra of the quantum-classical hybrid, thus excluding the other probabilistic theories, e. g. the PR-boxes.[21]

The existence of one (or more) purely mathematical postulate is a common feature of all previous attempts of operational axiomatization of QM, and it is hoped that the CJ postulate will be easier to derive from pure operational principles. Moreover, it will be illuminating to compare all different axiomatic frameworks, to see if from their union one can extract a set of purely operational axioms for QM.

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