Teleportation Protocols for Abstract State Spaces

Alex Wilce

Susquehanna University

Zurich, May 2008

(Joint work with H. Barnum, J. Barrett and M. Leifer)

- QM is (can be viewed as) a probability calculus with

Introduction

- QM is (can be viewed as) a probability calculus with
  - a relatively classical interpretation,

Introduction

- QM is (can be viewed as) a probability calculus with
  - a relatively classical interpretation,
  - a rather non-classical formal apparatus.

Introduction

- QM is (can be viewed as) a probability calculus with
  - a relatively classical interpretation,
  - a rather non-classical formal apparatus.

- To what extent can we motivate this apparatus in purely probabilistic or information-theoretic terms?

Introduction

- QM is (can be viewed as) a probability calculus with
  - a relatively classical interpretation,
  - a rather non-classical formal apparatus.

- To what extent can we motivate this apparatus in purely probabilistic or information-theoretic terms?
  - Old problem (von Neumann, Mackey, Ludwig...)

Introduction

- QM is (can be viewed as) a probability calculus with
  - a relatively classical interpretation,
  - a rather non-classical formal apparatus.
- To what extent can we motivate this apparatus in purely probabilistic or information-theoretic terms?
  - Old problem (von Neumann, Mackey, Ludwig...)
  - New input from QIT (Brassard-Fuchs, Hardy, D'Ariano, Joyal,...)

- To approach this question: look at *general* probabilistic theories, and ask: what's special about QM?

- To approach this question: look at *general* probabilistic theories, and ask: what's special about QM?
    - Examples: no-cloning, no-broadcasting theorems are quite generic (BBLW06, 07).

- To approach this question: look at *general* probabilistic theories, and ask: what's special about QM?
    - Examples: no-cloning, no-broadcasting theorems are quite generic (BBLW06, 07).
    - Teleportation *isn't* so generic!

Outline

(1) Abstract state spaces
(2) Composite systems
(3) Teleportation protocols

1. Abstract State Spaces

Definition

For purposes of this talk, an **abstract state space** is a pair (A, u_A) where

(i) A is a (finite-dimensional!) ordered real vector space with (closed, generating) positive cone A_+ of *un-normalized states*.

(ii) u_A is a *strictly* positive linear functional, called the *order unit*, picking out a set of *normalized states*

$$\Omega_A := u_A^{-1}(1).$$

Remarks:

- Ω_A is compact
- Any (f.d.) compact convex set has the form Ω_A for a canonical (A, u): take $A = \text{Aff}(\Omega)^*$, where $\text{Aff}(\Omega)$ is the space of real affine functionals on Ω, and set $u_A \equiv 1$.
- The convex hull of $\Omega \cup -\Omega$ is the unit ball for a norm, called the *base norm*, such that $\|\alpha\| = u(\alpha)$ for $\alpha \in A_+$.

Examples

Classical: $A = \mathbb{R}^X$, X a finite set with $u(f) = \sum_{x \in X} f(x)$; here Ω_A is the set of probability weights on X. *Note that A has this form iff Ω_A is a simplex.*

Quantum: $A = \mathscr{B}_h(\mathbf{H})$ = self-adjoint operators on complex (f.d.) Hilbert space **H** with $u(A) = \mathrm{Tr}(A)$; here Ω_A is the set of density operators.

Neither: A = $n \times n$ matrices with column sums = constant, with $u(a)$ = column sum; here Ω_A is the set of stochastic matrices. (In 2×2 case, a square.)

Note: Any abstract state space can be represented *concrete* state space $A(X, \mathfrak{A})$ where $(X, \mathfrak{A})$ is a *test space*.

Effects and Observables

Definition

An **effect** on an abstract state space A is a positive functional $a \in A^*$ with $a(\alpha) \leq 1$ for all $\alpha \in \Omega_A$. Write $[0, u_A]$ for the set of effects.

Interpretation: a represents an event – e.g., measurement outcome – with probability $a(\alpha)$ in state normalized state ω. Thus:

Definition

An (discrete) **observable** on A is a sequence $(a_1, ..., a_n)$ of effects with $\sum_i a_i = u_A$.

In classical examples, observables are (discrete) fuzzy random variables; in quantum examples, discrete POVMs.

Self-duality

Given an inner product on an abstract state space (A, u), we can define an **internal dual cone** by

$$A^+ = \{b \in A | \forall a \in A_+, \langle b, a \rangle \geq 0\}.$$

If $\langle \, , \, \rangle$ can be chosen so that $A^+ = A_+$, one says that A (or A_+) is **self-dual.** Finite-dimensional classical, and all quantum, state spaces are self-dual in this sense.

Theorem (Koechers, Vinberg)

Let A be an irreducible, finite-dimensional, self-dual state space. Suppose the group of affine automorphisms of A_+ acts transitively on the interior of A_+. Then Ω_A is affinely isomorphic to one of the following: (1) The set of density operators on an n-dimensional Hilbert space (i.e., A is quantum); (2) an n-ball; (3) the set of 3×3 trace-one matrices over the octonions.

Weak self-duality

A weaker condition is that there exist an *order-isomorphism* (a positive linear mapping with positive inverse)

$$\eta : A^* \simeq A.$$

If this is the case, we shall say that A is **weakly self-dual**.

Example: Let $A = \mathrm{Aff}(\Omega)$ where Ω is a square. There are just four minimal extremal effects, corresponding to the four faces of Ω; using these, one can easily construct the desired isomorphism, so this cone is weakly self-dual. It's not self-dual, however: V^+ is V_+ rotated by $\pi/4$.

2. Composite Systems

Suppose we want to model a composite system A comprising several sub-systems $A_1, ..., A_n$. We shall assume that a state ω of A is defined by a joint probability assignment

$$\omega : [0, u_1] \times \cdots \times [0, u_n] \to \mathbb{R}.$$

Such a state is **non-signaling** iff, for all observables E on A_1,

$$\omega_E(a_2, ..., a_n) := \sum_{a \in E} \omega(a, a_2, ..., a_n)$$

is independent of E, and similarly for the other components.

Theorem (KRF '87; JB '05)

ω is non-signaling iff it extends to an n-linear form on $A_1^* \times \cdots \times A_n^*$.

Identify $\bigotimes_i A_i$ with the space of n-linear forms on $A_1^* \times \cdots \times A_n^*$. Thus, if $\alpha_i \in A_i$ for $i = 1, ..., n$, the pure tensor $\otimes_i \alpha_i$ is the form

$$(\otimes_i \alpha_i)(a_1, ..., a_n) = \Pi_i \alpha_i(a_i).$$

Call a form $\omega \in \bigotimes_i A_i$ *positive* iff

$$a_1, ..., a_n \geq 0 \;\Rightarrow\; \omega(a_1, ..., a_n) \geq 0$$

for all $a_i \in A_i^*$. Example: $\otimes_i \alpha_i$ with $\alpha_1, ..., \alpha_n \geq 0$.

Definition

A **composite** of $A_1, ..., A_n$ is a state space consisting of n-linear forms on $A_1^* \times \cdots \times A_n^*$, ordered by a cone of positive forms containing all pure tensors, and with order unit $u = \otimes_i u_i$.

(Thus, if we ignore the ordered structure, a composite A of $A_1, ..., A_n$ is just $A_1 \otimes \cdots \otimes A_n$.)

Bipartite composites

Examples

Bipartite composites

Examples

- The **maximal tensor product**, $A \otimes_{\max} B$, uses the cone of all positive forms;

Bipartite composites

Examples

- The **maximal tensor product**, $A \otimes_{\max} B$, uses the cone of all positive forms;
- The **minimal tensor product**, $A \otimes_{\min} B$, uses the cone generated by the pure tensors.

Bipartite composites

Examples

- The **maximal tensor product**, $A \otimes_{\max} B$, uses the cone of all positive forms;
- The **minimal tensor product**, $A \otimes_{\min} B$, uses the cone generated by the pure tensors.
- If A, B are quantum state spaces, the usual cone of bipartite quantum states is properly between the maximal and minimal cones in $A \otimes B$.

Entanglement

Definition

States of $A \otimes_{\text{max}} B$ not in $A \otimes_{\text{min}} B$ are **entangled**.

Dually, entangled *effects* are those in $(A \otimes_{\text{min}} B)^*$ not in $(A \otimes_{\text{max}} B)^*$.

Entanglement

Definition

States of $A \otimes_{\text{max}} B$ not in $A \otimes_{\text{min}} B$ are **entangled**.

Dually, entangled *effects* are those in $(A \otimes_{\text{min}} B)^*$ not in $(A \otimes_{\text{max}} B)^*$.

Theorem (Namioka-Phelps)

$A \otimes_{max} B = A \otimes_{min} B$ for all B iff A is classical.

Entanglement

Definition

States of $A \otimes_{\max} B$ not in $A \otimes_{\min} B$ are **entangled**.

Dually, entangled *effects* are those in $(A \otimes_{\min} B)^*$ not in $(A \otimes_{\max} B)^*$.

Theorem (Namioka-Phelps)

$A \otimes_{max} B = A \otimes_{min} B$ for all B iff A is classical.

Thus, entanglement is a feature of any theory involving more than one non-classical state space – unless artificially ruled out by stubborn insistence on using $\otimes_{\min}$.

Marginal and Conditional States

Any state ω in a composite AB has *marginal* or *reduced* states $\omega_A \in A$, $\omega_B \in B$, given by

$$\omega_A(a) := \omega(a, u_B) \text{ and } \omega_B(b) = \omega(u_A, b).$$

If $\omega_A(a) \neq 0$, the **conditional state** of B given effect $a \in A^*$ is given by

$$\omega_{B|a}(b) := \omega(a, b)/\omega_A(a)$$

Just as in QM, pure entangled states have mixed marginals:

Lemma

Let ω be a pure state in $A \otimes B$. If either ω_A or ω_B is pure, then $\omega = \omega_A \otimes \omega_B$.

Bipartite states as operators

Every bipartite state ω in a composite AB corresponds to a positive operator $\hat{\omega} : A^* \to B$, given by

$$\hat{\omega}(a) = \omega(a, \cdot\,).$$

Any positive operator $\phi : A^* \to B$ with $\phi(u) \in \Omega_B$ has the form $\hat{\omega}$ for a state $\omega \in A \otimes_{\max} B$. Note that $\hat{\omega}(u_A) = \omega_B$; thus, $\hat{\omega}(a)$ is the un-normalized *conditional* state of B given the effect a on A.

Similarly, a bipartite effect $f \in (AB)^*$ corresponds to an operator $\hat{f} : A \to B^*$, given by

$$\hat{f}(\alpha)(\beta) = f(\alpha \otimes \beta)$$

for all $\alpha \in A$ and $\beta \in B$.

N-partite composites

The theory of composites of $n > 2$ systems is a bit more delicate.

N-partite composites

The theory of composites of $n > 2$ systems is a bit more delicate.

Examples: Let A_1, A_2, A_3 be (say) quantum state spaces. Some composites:

N-partite composites

The theory of composites of $n > 2$ systems is a bit more delicate.

Examples: Let A_1, A_2, A_3 be (say) quantum state spaces. Some composites:

- $A_1 \otimes_{\text{min}} A_2 \otimes_{\text{min}} A_3$

N-partite composites

The theory of composites of $n > 2$ systems is a bit more delicate.

Examples: Let A_1, A_2, A_3 be (say) quantum state spaces. Some composites:

- $A_1 \otimes_{\text{min}} A_2 \otimes_{\text{min}} A_3$
- $A_1 \otimes_{\text{max}} A_2 \otimes_{\text{max}} A_3$

N-partite composites

The theory of composites of $n > 2$ systems is a bit more delicate.

Examples: Let A_1, A_2, A_3 be (say) quantum state spaces. Some composites:

- $A_1 \otimes_{\text{min}} A_2 \otimes_{\text{min}} A_3$
- $A_1 \otimes_{\text{max}} A_2 \otimes_{\text{max}} A_3$
- $A_1 \otimes_{\text{QM}} A_2 \otimes_{\text{QM}} A_3$

N-partite composites

The theory of composites of $n > 2$ systems is a bit more delicate.

Examples: Let A_1, A_2, A_3 be (say) quantum state spaces. Some composites:

- $A_1 \otimes_{\text{min}} A_2 \otimes_{\text{min}} A_3$
- $A_1 \otimes_{\text{max}} A_2 \otimes_{\text{max}} A_3$
- $A_1 \otimes_{\text{QM}} A_2 \otimes_{\text{QM}} A_3$
- $A_1 \otimes_{\text{min}} (A_2 \otimes_{\text{QM}} A_3)$

N-partite composites

The theory of composites of $n > 2$ systems is a bit more delicate.

Examples: Let A_1, A_2, A_3 be (say) quantum state spaces. Some composites:

- $A_1 \otimes_{\text{min}} A_2 \otimes_{\text{min}} A_3$
- $A_1 \otimes_{\text{max}} A_2 \otimes_{\text{max}} A_3$
- $A_1 \otimes_{\text{QM}} A_2 \otimes_{\text{QM}} A_3$
- $A_1 \otimes_{\text{min}} (A_2 \otimes_{\text{QM}} A_3)$
- $(A_1 \otimes_{\text{min}} A_2) \otimes_{\text{max}} A_3$...

N-partite composites

The theory of composites of $n > 2$ systems is a bit more delicate.

Examples: Let A_1, A_2, A_3 be (say) quantum state spaces. Some composites:

- $A_1 \otimes_{\text{min}} A_2 \otimes_{\text{min}} A_3$
- $A_1 \otimes_{\text{max}} A_2 \otimes_{\text{max}} A_3$
- $A_1 \otimes_{\text{QM}} A_2 \otimes_{\text{QM}} A_3$
- $A_1 \otimes_{\text{min}} (A_2 \otimes_{\text{QM}} A_3)$
- $(A_1 \otimes_{\text{min}} A_2) \otimes_{\text{max}} A_3$...

etc!

Subsystems

If A is a composite of $A_1, ..., A_n$, then given $J \subseteq \{1, ..., n\}$ and a list $a = (a_i)_{i \notin J}$ of functionals $a_i \in A_i^*$ for $i \in I \setminus J$, we can define a *partially evaluated* form

$$\omega_J(a) \in \bigotimes_{j \in J} A_j.$$

This represents an un-normalized *conditional* state.

Example: For $n = 4$,

$$\omega_{1,3}(a_2, a_4) : (a_1, a_3) \mapsto \omega(a_1, a_2, a_3, a_4).$$

Definition (Subsystems)

Let A be a composite of $A_1, ..., A_n$, and suppose $J \subseteq \{1, ..., n\}$. The *J-reduced subsystem* of A is $\bigotimes_{j \in J} A_j$, ordered by the cone generated by the partially evaluated states $\omega_J(f)$.

Regular composites

Definition (Regularity)

We say that A is a **regular** composite of $A_1, ..., A_n$ iff, for all $J \subseteq \{1, ..., n\}$, A is a composite of A_J and $A_{I \setminus J}$. Equivalently:

Regular composites

Definition (Regularity)

We say that A is a **regular** composite of $A_1, ..., A_n$ iff, for all $J \subseteq \{1, ..., n\}$, A is a composite of A_J and $A_{I \setminus J}$. Equivalently:

(a) $\mu \in A_J$, $\nu \in A_{I \setminus J} \Rightarrow \mu \otimes \nu \in A$;

Regular composites

Definition (Regularity)

We say that A is a **regular** composite of $A_1, ..., A_n$ iff, for all $J \subseteq \{1, ..., n\}$, A is a composite of A_J and $A_{I \setminus J}$. Equivalently:

(a) $\mu \in A_J$, $\nu \in A_{I \setminus J} \Rightarrow \mu \otimes \nu \in A$;

(b) $f \in A_J^*$, $g \in A_{I \setminus J}^*$ implies $f \otimes g \in A^*$.

Regular composites

Definition (Regularity)

We say that A is a **regular** composite of $A_1, ..., A_n$ iff, for all $J \subseteq \{1, ..., n\}$, A is a composite of A_J and $A_{I \setminus J}$. Equivalently:

(a) $\mu \in A_J$, $\nu \in A_{I \setminus J} \Rightarrow \mu \otimes \nu \in A$;

(b) $f \in A_J^*$, $g \in A_{I \setminus J}^*$ implies $f \otimes g \in A^*$.

Examples:

Regular composites

Definition (Regularity)

We say that A is a **regular** composite of $A_1, ..., A_n$ iff, for all $J \subseteq \{1, ..., n\}$, A is a composite of A_J and $A_{I \setminus J}$. Equivalently:

(a) $\mu \in A_J$, $\nu \in A_{I \setminus J} \Rightarrow \mu \otimes \nu \in A$;

(b) $f \in A_J^*$, $g \in A_{I \setminus J}^*$ implies $f \otimes g \in A^*$.

Examples:

- Any bipartite composite (trivially)

Regular composites

Definition (Regularity)

We say that A is a **regular** composite of $A_1, ..., A_n$ iff, for all $J \subseteq \{1, ..., n\}$, A is a composite of A_J and $A_{I \setminus J}$. Equivalently:

(a) $\mu \in A_J$, $\nu \in A_{I \setminus J} \Rightarrow \mu \otimes \nu \in A$;

(b) $f \in A_J^*$, $g \in A_{I \setminus J}^*$ implies $f \otimes g \in A^*$.

Examples:

- Any bipartite composite (trivially)
- $A_1 \otimes \cdots \otimes A_n$, $\otimes$ any monoidal product

Regular composites

Definition (Regularity)

We say that A is a **regular** composite of $A_1,...,A_n$ iff, for all $J \subseteq \{1,...,n\}$, A is a composite of A_J and $A_{I\setminus J}$. Equivalently:

(a) $\mu \in A_J$, $\nu \in A_{I\setminus J} \Rightarrow \mu \otimes \nu \in A$;

(b) $f \in A_J^*$, $g \in A_{I\setminus J}^*$ implies $f \otimes g \in A^*$.

Examples:

- Any bipartite composite (trivially)
- $A_1 \otimes \cdots \otimes A_n$, $\otimes$ any monoidal product
- $A \otimes_{\text{min}} (B \otimes_{\text{max}} C)$, $(A \otimes_{\text{min}} B) \otimes_{\text{max}} C$;

Regular composites

Definition (Regularity)

We say that A is a **regular** composite of $A_1, ..., A_n$ iff, for all $J \subseteq \{1, ..., n\}$, A is a composite of A_J and $A_{I\setminus J}$. Equivalently:

(a) $\mu \in A_J$, $\nu \in A_{I\setminus J} \Rightarrow \mu \otimes \nu \in A$;

(b) $f \in A_J^*$, $g \in A_{I\setminus J}^*$ implies $f \otimes g \in A^*$.

Examples:

- Any bipartite composite (trivially)
- $A_1 \otimes \cdots \otimes A_n$, $\otimes$ any monoidal product
- $A \otimes_{\text{min}} (B \otimes_{\text{max}} C)$, $(A \otimes_{\text{min}} B) \otimes_{\text{max}} C$;

Non-example: $(A \otimes_{\text{min}} A) \otimes_{\text{max}} (A \otimes_{\text{min}} A)$ where A is any weakly self-dual nonclassical state space.

3. Teleportation

As observed above, if ω is a bipartite state on AB, with corresponding operator $\hat{\omega} : A^* \rightarrow B$, then $\hat{\omega}(a) \in B_+$ represents the un-normalized conditional state of B given measurement result a.

Lemma (Remote Evaluation)

Let ABC be a regular composite of A, B and C with reduced systems AB and BC. If $f \in (AB)^*$ is a bipartite effect and $\omega \in BC$ is a bipartite state, then for any state $\alpha \in A$,

$$(\alpha \otimes \omega)(f \otimes -) = \hat{\omega}(\hat{f}(\alpha)).$$

If the tripartite system ABC is in state $\alpha \otimes \omega$, α unknown, then conditional on securing measurement outcome f on AB, the state of C is a *known function of* α.

Conclusive teleportation

If $C = A$ and $\tau = \hat{\omega} \circ \hat{f}$ is *physically reversible* (invertible with norm non-increasing inverse), then performing the operation τ^{-1} at C reproduces α. This is *conclusive (one-outcome post-selected) teleportation*. When this is possible, we say that *B teleports A*.

Theorem (Conclusive TP)

B teleports A iff there exist a positive embedding $i : A \to B^*$, and a positive idempotent compression $P : B^* \to B^*$ with range $i(A)$.

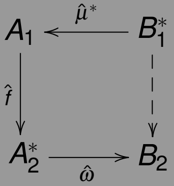
Entanglement Swapping

Remote evaluation is a special case of a more general result:

Theorem (State Pivoting)

Let $A = A_1 A_2$ and $B = B_1 B_2$ be composite systems, and let AB be a regular composite of A_1, A_2, B_1 and B_2. If μ is a state of $A_1 B_1$ and ω is a state of $A_2 B_2$, then for any $f \in A^*$,

$$\hat{\omega} \circ \hat{f} \circ \hat{\mu}^* = (\mu \otimes \omega)_B(f) \in B.$$



- The protocol is simply

$$\mu \mapsto (\mu \otimes \omega)(f \otimes \; - \;)$$

- The protocol is simply

$$\mu \mapsto (\mu \otimes \omega)(f \otimes \, - \,)$$

- If ω and f realize a conclusive teleportation protocol, we end up with state μ pivoted from $A_1 B_1$ to $B_1 B_2 = B$.

- The protocol is simply

$$\mu \mapsto (\mu \otimes \omega)(f \otimes -)$$

- If ω and f realize a conclusive teleportation protocol, we end up with state μ pivoted from $A_1 B_1$ to $B_1 B_2 = B$.
- Therefore, if $A_1 \simeq B_2$, we need $A_1 B_1 \simeq B_1 B_2$. This is what fails for $(A \otimes_{\text{min}} A) \otimes_{\text{max}} (A \otimes_{\text{min}} A)$ with A weakly self-dual.

Deterministic Teleportation

In order to *deterministically* teleport an unknown state $\alpha \in A$ through B, we need not just one entangled effect f, but an entire observable's worth.

Definition

A *deterministic teleportation protocol* for A through B consists of an observable $E = (f_1, ..., f_n)$ on AB and a state ω in BA, such that for all $i = 1, ..., n$, the operator $\hat{f}_i \circ \hat{\omega}$ is physically invertible.

Theorem

Suppose that G is a finite group acting linearly on A in such a way as to preserve Ω. Suppose that

(a) *there exists a unique G-invariant state $\alpha_o \in \Omega$, and*

(b) *there exists an order-automorphism $\hat{\omega} : A^* \to A$ with $\hat{\omega}(u) = \alpha_o$.*

Then $A \otimes_{min} (A \otimes_{max} A)$ supports a deterministic teleportation protocol.

Sketch of proof: Not that $\hat{\omega}$ defines a bipartite state $\omega \in A \otimes_{\max} A$. For each $g \in G$, let $f_g \in (A \otimes_{\max} B)^*$ correspond to the operator

$$\hat{f}_g = \frac{1}{|G|} \hat{\omega}^{-1} \circ g.$$

Then $E = \{f_g | g \in G\}$ is an observable, and (E, ω) is a deterministic teleportation protocol. $\square$

Example

Let $A = \text{Aff}(\Omega)^*$ with Ω a square. We've seen that this is weakly self-dual. Let $G = D_4$ acting on Ω in the obvious way: the center of the square is the unique fixed point. It's easy to see that the obvious isomorphism $A^* \simeq A$ (suitably normalized) takes u to the center of the square. Thus, $A \otimes_{\text{max}} A$ supports deterministic teleportation.

Conclusions

Conclusions

The possibility of teleportation is a sharp constraint on physical theories; however

Conclusions

The possibility of teleportation is a sharp constraint on physical theories; however

There do exist non-classical, non-quantum theories supporting deterministic TP.

References

References

BBLW06: Cloning and broadcasting in general probabilistic theories, quant-ph/061129

References

BBLW06: Cloning and broadcasting in general probabilistic theories, quant-ph/061129

BBLW07 (A) general no-cloning theorem, Phys. Rev. Lett. **99** (1977) 240501; arXiv:0707.0620.

References

BBLW06: Cloning and broadcasting in general probabilistic theories, quant-ph/061129

BBLW07 (A) general no-cloning theorem, Phys. Rev. Lett. **99** (1977) 240501; arXiv:0707.0620.

BBLW08 Teleportation in general probabilistic theories, arXiv: ...