THE SYMMETRIC PRODUCT AND MOONSHINE IN THE HETEROTIC STRING

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INTRODUCTION AND MOTIVATION

- Consider the Elliptic genus of $K3$.

$$F(K3;T,V)=$$
$$\mathrm{Tr}_{RR}\left((-1)^{F^{K3}+\bar{F}^{K3}}e^{2\pi iVF^{K3}}e^{2\pi iT(L_0-c/24)}\bar{e}^{-2\pi i\bar{T}(\bar{L}_0-c/24)}\right)$$
$$=\sum_{m\geq 0,l}c(4m-l^2)e^{2\pi imT}e^{2\pi ilV}$$

The trace is taken over the Ramond sector.

The elliptic genus is holomorphic in T,V.

The generating function for the elliptic genus of the symmetric product of $K3$ is given by

Moore, Dijkgraaf , Verlinde, Verlinde (1995)

$$
\begin{aligned}
G(U,T,V) &= \sum_{N=0}^{\infty} e^{2\pi i N U} F(K3^N/N;T,V) \\
&= \prod_{n>0, m\geq 0, l\in\mathbb{Z}} \frac{1}{(1-e^{2\pi i(nU+mT+lV)})^{c(4nm-l^2)}}
\end{aligned}
$$

- The symmetric product G is closely associated to Φ_{10}

$$\Phi_{10}(U,T,V) = e^{-2\pi i(U+T+V)} \frac{1}{(1-e^{-2\pi i V})^2} \times$$

$$\prod_{m=1}^{\infty} \frac{1}{(1-e^{2\pi i m T})^{20}(1-e^{2\pi i(mT+V)})^2(1-e^{2\pi i(nT-V)})^2} \times$$

$$G(U,T,V)$$

Essentially the additional terms complete the product

$$\Phi_{10}(U,T,V) = e^{-2\pi i(U+T+V)} \times$$

$$= \prod_{n\geq 0, m\geq 0, l\in\mathbb{Z}; n=m=0, l<0} \frac{1}{(1-e^{2\pi i(nU+mT+lV)})^{c(4nm-l^2)}}$$

Φ_{10} is the unique Siegal modular form of weight 10 under the group $Sp(2,\mathbb{Z}) \sim SO(3,2;\mathbb{Z})$.

Also called the Igusa cusp form.

Φ_{10} the modular form associated with the elliptic genus of $K3$.

Modular properties:

Arrange the parameters as

$$\Omega = \begin{pmatrix} U & V \\ V & T \end{pmatrix}$$

Then

$$\Phi_{10}((C\Omega + D)^{-1}(A\Omega + B)) = [\det(C\Omega + D)]^{10}\Phi_{10}(\Omega)$$

where

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix}^T \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}_{4\times 4} \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

A, B, C, D are 2×2 matrices with integer elements.

This modular property is analogous to that of the Dedekind η function

$$\eta(\tau) = e^{\pi i\tau/12}\prod_{n=1}^{\infty}(1 - e^{2\pi i\tau n})$$

we have the modular property

$$\eta^{24}[(c\tau + d)^{-1}(a\tau + d)] = (c\tau + d)^{12}\eta^{24}(\tau)$$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathbb{Z})$$

• Generalization of Siegel modular forms associated with the twisted elliptic genus of $K3$ are known.

There exists $\mathbb{Z}_N$ quotients of $K3$ for which the Hodge diamond of $K3/\mathbb{Z}_N$ becomes

$$h_{(0,0)} = h_{(2,2)} = h_{(0,2)} = h_{(2,0)} = 1,$$
$$h_{(1,1)} = 2\left(\frac{24}{N+1} - 2\right) = 2k$$

N	$h_{(1,1)}$	k
1	20	10
2	12	6
3	8	4
5	4	2
7	2	1

• Let g' be action of this quotient,
the twisted elliptic genus of $K3$ is defined as

$$
\begin{aligned}
& F^{(r,s)}(T,V) \\
= \; & \frac{1}{N}\mathrm{Tr}^{K3}_{RR;g'^r}\left((-1)^{F^{K3}+\bar{F}^{K3}} g'^s e^{2\pi i V F^{K3}} e^{2\pi i T(L_0-c/24)} \bar{q}^{-2\pi i \bar{T}(\bar{L}_0-c/24)}\right) \\
& \qquad 0 \leq r, s, \leq (N-1).
\end{aligned}
$$

Associated with this twisted elliptic genus there exists a Siegal modular form of weight k

$$\Phi_k(U,T,V)$$

• There is a similar construction of this modular form that proceeds by taking the symmetric product of the twisted elliptic genus of $K3$.

• Modular forms like the Dedekind $\eta(\tau)$ function appear in effective actions of string compactifications.

Usually the τ parameter is replaced by some compactificaton moduli.

eg. The coefficient of the Gauss-Bonnet term of type II on $K3 \times T^2$

$$R^2 \ln(|\eta(T)|^{24} T_2^6)$$

R^2 is the Gauss-Bonnet curvature.

T_2 is the imaginary part of T , the Kähler modulus of the torus.

• Does Siegel modular forms $\Phi_k(U, T, V)$ appear in string effective actions with U, T, V being some moduli of the compactification.

• The weight of the Siegel modular form captures the information of the Hodge number $h_{(1,1)}$ of the quotient of $K3$.

Are there more detailed information captured in the effective action?

eg. Hints of M_{24} symmetry in the effective action ?

SUMMARY OF THE RESULTS

- Consider the Heterotic $E_8 \times E_8$ string on $(K3 \times T^2)/\mathbb{Z}_N$.

Z_N acts as the quotient mentioned before on $K3$ together with a shift of unit $1/N$ along one of the S^1 of T^2.

We call this orbifold the CHL orbifold of $K3$.

To ensure supersymmetry embed the spin connection of $K3$ into the gauge connection.

These models have $N = 2$ supersymmetry in $d = 4$.

• In the standard embedding when $SU(2)$ from one of the E_8 is set equal to the spin connection the gauge symmetry is broken to $E_7 \times E_8$

Consider 1-loop corrections to the gauge couplings

$$\frac{1}{g^2(E_7)} = \Delta_{G'}(T, U, V), \qquad \frac{1}{g^2(E_8)} = \Delta_{G'}(T, U, V)$$

which depend on the Kähler and complex structure moduli T, U of the torus T^2.

We also turn on the Wilson line

$$V = A_1 + iA_2$$

with values in say a $U(1)$ of the unbroken E_8.

- We show that the difference in one loop threshold corrections

$$\Delta_G(T,U,V) - \Delta_{G'}(T,U,V) = -48\log\left[(\det \mathrm{Im}\Omega)^k \left|\Phi_k(T,U,V)\right|^2\right],$$

where Ω^k is a weight k modular form transforming under subgroups of $Sp(2,\mathbb{Z})$ with k

$$k = \frac{24}{N+1} - 2\,,$$

where $N = 2,3,5,7$ labels the various CHL orbifolds.

• The precursor to evaluating the one loop threshold corrections is the New supersymmetric index

$$\mathcal{Z}_{\rm new}(q,\bar{q}) = \frac{1}{\eta^2(\tau)} \mathrm{Tr}_R \left(F e^{i\pi F} q^{L_0 - \frac{c}{24}} \bar{q}^{\bar{L}_0 - \frac{\bar{c}}{24}} \right) \ .$$

We take left movers to be bosonic and right movers to be super symmetric in the heterotic string.

The trace in the above expression is taken over the Ramond sector in the internal CFT with central charges $(c, \bar{c}) = (22, 9)$.

F is the world sheet fermion number of the right moving $\mathcal{N} = 2$ supersymmetric internal CFT.

• For the $K3 \times T^2$ compactification the new supersymmetric index is given by Harvey, Moore (1995)

$$\mathcal{Z}_{\rm new}(q,\bar{q}) = -8\,\frac{E_4(q)E_6(q)}{\eta(q)^{24}}\,\Gamma_{2,2}(q,\bar{q})$$

$\Gamma_{2,2}$ is the lattice sum of momenta and winding over the T^2.

$E_4(q)$ is from the unbroken E_8 lattice.

$E_6(q)$ is from the E_7 lattice together with the $K3$.

- The part which has its orgin due to the $K3$

$$\frac{E_6(q)}{\eta(q)^{12}}$$

admits a q expansion which can be organized as sums of irreducible representations of the Mathieu group M_{24}.
Cheng, Dong, Duncan, Harvey, Kachru Wrase (2013)

• We evaluate the new supersymmetric index for the CHL $K3 \times T^2/\mathbb{Z}_N$ orbifold compactifications of the heterotic string.

We show that it is related to the twisted index of $K3$.

It admits a decomposition in terms of the Mackay-Thompson Series associated with the $\mathbb{Z}_N$ action g' embedded in M_{24}.

The Mackay-Thompson series is essentially

$$\mathrm{Tr}(g')$$

over various representations of M_{24}.

SPECTRUM OF HETEROTIC ON THE CHL ORBIFOLD OF $K3$

- The orbifold $(K3 \times T^2)/\mathbb{Z}_N$ preserves the $SU(2)$ holonomy.

It preserves the $SU(2)$ invariant $(0,2)$ and $(2,0)$-forms.

We can organize the multiplets in terms of $\mathcal{N}=2$ multiplets in $d=4$.

• The gravity multiplet in $d = 10$ dimensionally reduces to a $\mathcal{N} = 2$ gravity multiplet in $d = 4$.
+ 3 $\mathcal{N} = 2$ vector multiplets + $2k$ Hypermultiplets.

$$R(10) \to R(4) + 3V(4) + 2kH(4)\,.$$

The 3 vectors arise from

$$g_{\mu i}, \qquad B_{\mu i}$$

i labels to the 2 directions along the torus.

One of the vectors forms part of the $d = 4$ gravity multiplet,

the the rest forms the 3 vector multiplets.

The number of hypers depend on k:

Note the number of scalars from depend on the $2k$ $(1,1)$ forms. on which anti-symmetric tensor B_{MN} can be reduced.

• Proceeding similarly after embedding the spin connection into one of the E_8

$$\int_{K3} \mathrm{Tr}(F \wedge F) = \int_{K3} \mathrm{Tr}(R \wedge R)$$

The gauge group breaks to $E_7 \times E_8$.

• The Yang-Mills multiplet in $d = 10$ reduces to

$$\begin{aligned} Y(10) \rightarrow \quad & V(4)[(\mathbf{133}, \mathbf{1}) + (\mathbf{1}, \mathbf{248})] \\ & + H(4)[k(\mathbf{56}, \mathbf{1}) + (4(k+2) - 3)(\mathbf{1}, \mathbf{1})] \,. \end{aligned}$$

Here we have kept track of the $E_7 \times E_8$ representations.

• The CHL orbifolding only affects the number of hypers in the spectrum.

It leaves the vectors invariant.

• The classical moduli space of the vector multiplets is not affected.

However the threshold corrections will show:

T-duality group of the orbifolded theory are sub-groups of the parent theory.

THE NEW SUPERSYMMETRIC INDEX

- The new supersymmetric index is defined by

$$\mathcal{Z}_{\rm new}(q,\bar{q}) = \frac{1}{\eta^2(\tau)} \mathrm{Tr}_R \left(F e^{i\pi F} q^{L_0 - \frac{c}{24}} \bar{q}^{\bar{L}_0 - \frac{\bar{c}}{24}} \right) .$$

The trace is taken over the internal CFT with central charge $(c, \tilde{c}) = (22, 9)$.

The left movers are bosonic while the right movers are supersymmetric.

The right moving internal CFT has a $\mathcal{N} = 2$ superconformal symmetry.

It admits a $U(1)$ current which can serve as the world sheet fermion number , we denote this as F.

The subscript R refers to the fact that we take the trace in the Ramond sector for the right movers.

• The index can be explicitly evaluated for the $N = 2$ CHL orbifold $K3 \times T^2/\mathbb{Z}_2$.

This orbifold is realized as

$$
\begin{aligned}
g: \quad & (y^4, y^5, y^6, y^7, y^8, y^9) \to (y^4, y^5, -y^6, -y^7, -y^8, -y^9), \\
g': \quad & (y^4, y^5, y^6, y^7, y^8, y^9) \to (y^4 + \pi, y^5, y^6 + \pi, y^7, y^8, y^9).
\end{aligned}
$$

The g action realizes $K3$ as a Z_2 orbifold,

while g' implements the CHL orbifold.

This orbifold is coupled to a $1/2$ shift in the E_8' lattice.

- We define the lattice momenta on the T^2 which is given by

$$\frac{1}{2}p_R^2 = \frac{1}{2T_2U_2}|-m_1U+m_2+n_1T+n_2TU|^2,$$
$$\frac{1}{2}p_L^2 = \frac{1}{2}p_R^2 + m_1n_1 + m_2n_2\,.$$

The variables T, U refer to the Kähler moduli and the complex structure of the torus T^2.

The lattice sums over T^2 are

$$\Gamma_{2,2}^{(0,0)}(\tau,\bar\tau) = \sum_{m_1,m_2,n_1,n_2\in\mathbb{Z}} q^{\frac{p_L^2}{2}}\,\bar q^{\frac{p_R^2}{2}},$$

$$\Gamma_{2,2}^{(0,1)}(\tau,\bar\tau) = \sum_{m_1,m_2,n_1,n_2\in\mathbb{Z}} q^{\frac{p_L^2}{2}}\,\bar q^{\frac{p_R^2}{2}}(-1)^{m_1},$$

$$\Gamma_{2,2}^{(1,0)}(\tau,\bar\tau) = \sum_{\substack{m_1,m_2,n_2\in\mathbb{Z},\\ n_1\in\mathbb{Z}+\frac{1}{2}}} q^{\frac{p_L^2}{2}}\,\bar q^{\frac{p_R^2}{2}},$$

$$\Gamma_{2,2}^{(1,1)}(\tau,\bar\tau) = \sum_{\substack{m_1,m_2,n_2\in\mathbb{Z},\\ n_1\in\mathbb{Z}+\frac{1}{2}}} q^{\frac{p_L^2}{2}}\,\bar q^{\frac{p_R^2}{2}}(-1)^{m_1}.$$

- Evaluating the new supersymmetric index results in

$$\mathcal{Z}^{(2)}_{\rm new}(q,\bar{q}) = -\frac{2E_4}{\eta^{12}} \times \left[\frac{1}{\eta^{12}} \left\{ \Gamma^{(0,0)}_{2,2} 2E_6 + \Gamma^{(0,1)}_{2,2} \frac{2}{3} \left(E_6 + 2\mathcal{E}_2(\tau) E_4 \right) \right\} \right.$$
$$\left. + \frac{2}{3\eta^{12}} \left\{ \Gamma^{(1,0)}_{2,2} \left(E_6 - \mathcal{E}_2(\tfrac{\tau}{2}) E_4 \right) + \Gamma^{(1,1)}_{2,2} \left(E_6 - \mathcal{E}_2(\tfrac{\tau+1}{2}) E_4 \right) \right\} \right].$$

$$\mathcal{E}_N(\tau) = \frac{12i}{\pi(N-1)} \partial_\tau \log \frac{\eta(\tau)}{\eta(N\tau)}.$$

• An important property of the new supersymmetric index

it can be written in terms of the twisted elliptic index of $K3$.

$$F^{(r,s)}(\tau,z) = \frac{1}{N}\mathrm{Tr}^{K3}_{RR;g'^r}\left((-1)^{F^{K3}+\bar{F}^{K3}} g'^s e^{2\pi i z F^{K3}} q^{L_0-c/24}\bar{q}^{\bar{L}_0-c/24}\right), \qquad 0\leq r,s,\leq (N-1).$$

For the $N=2$ orbifold the twisted indices are

$$F^{(0,0)}(\tau,z) = 4\left[\frac{\theta_2(\tau,z)^2}{\theta_2(\tau,0)^2}+\frac{\theta_3(\tau,z)^2}{\theta_3(\tau,0)^2}+\frac{\theta_4(\tau,z)^2}{\theta_4(\tau,0)^2}\right],$$

$$F^{(0,1)}(\tau,z) = 4\frac{\theta_2(\tau,z)^2}{\theta_2(\tau,0)^2}, \quad F^{(1,0)}(\tau,z) = 4\frac{\theta_4(\tau,z)^2}{\theta_4(\tau,0)^2},$$

$$F^{(1,1)}(\tau,z) = 4\frac{\theta_3(\tau,z)^2}{\theta_3(\tau,0)^2}.$$

• Using these expressions for the twisted elliptic genus it can be shown new supersymmetric index can be written as

$$\mathcal{Z}_{new}(q,\bar{q})^{(2)} = \frac{2E_4}{\eta^{12}} \times \sum_{a,b=0}^{1}$$

$$\left[\Gamma_{2,2}^{(a,b)} \left(\frac{\theta_2^6}{\eta^6} F^{(a,b)}(\tau, \tfrac{1}{2}) + q^{1/4} \frac{\theta_3^6}{\eta^6} F^{(a,b)}(\tau, \tfrac{1+\tau}{2}) - q^{1/4} \frac{\theta_4^6}{\eta^6} F^{(a,b)}(\tau, \tfrac{\tau}{2}) \right)\right]$$

MATHIEU MOONSHINE

• The analysis of the new supersymmetric index for CHL orbifolds of $K3$ shows it is essentially determined by the twisted elliptic index of $K3$.

It is known that the twisted elliptic genus of $K3$ admits M_{24} symmetry.

Therefore, it must be possible to discover the M_{24} representations in the new supersymmetric index for the CHL orbifolds of $K3$,

just as it was done for the new supersymmetric index for $K3$ compactifications.

Cheng, Dong, Duncan, Harvey, Kachru Wrase (2013)

• Recall how Mathieu moonshine – i.e. M_{24} representations

is seen in the elliptic genus of $K3$.

$$Z_{K3}(\tau, z) = 8\left[\frac{\theta_2(\tau,z)^2}{\theta_2(\tau,0)^2} + \frac{\theta_3(\tau,z)^2}{\theta_3(\tau,0)^2} + \frac{\theta_4(\tau,z)^2}{\theta_4(\tau,0)^2}\right]$$

Decompose the elliptic genus into the elliptic genera of the short and the long representations of the $\mathcal{N}=4$ super conformal algebra.

$$\begin{aligned} \mathrm{ch}_{h=\frac{1}{4},l=0}(\tau,z) &= -i\frac{e^{\pi i z}\theta_1(\tau,z)}{\eta(\tau)^3}\sum_{n=-\infty}^{\infty}\frac{e^{\pi i\tau n(n+1)}e^{2\pi i(n+\frac{1}{2})}}{1-e^{2\pi i(n\tau+z)}}, \\ \mathrm{ch}_{h=n+\frac{1}{4},l=\frac{1}{2}}(\tau,z) &= e^{2\pi i\tau(n-\frac{1}{8})}\frac{\theta_1(\tau,z)^2}{\eta(\tau)^2}. \end{aligned}$$

Then it can be seen

$$Z_{K3}(\tau, z) = 24\mathrm{ch}_{h=\frac{1}{4},l=0}(\tau, z) + \sum_{n=0}^{\infty} A_n^{(1)} \mathrm{ch}_{h=n+\frac{1}{4},l=\frac{1}{2}}(\tau, z).$$

where the first few values of $A_n^{(1)}$ are given by

$$A_n^{(1)} = -2, 90, 462, 1540, 4554, 11592, \ldots$$

These coefficients are either the dimensions or the sums of dimensions of the irreducible representations of the group M_{24}.
Eguchi, Ooguri, Tachikawa (2010)

- The twisted elliptic index $F^{(01)}$ admits the decomposition

$$2F^{(0,1)}(\tau, z) = 8\mathrm{ch}_{h=\frac{1}{4}, l=0}(\tau, z) + \sum_{n=0}^{\infty} A_n^{(2)} \mathrm{ch}_{h=n+\frac{1}{4}, l=\frac{1}{2}}(\tau, z).$$

the first few values of $A_n^{(2)}$ are given by

$$A_n^{(2)} = -2, -6, 14, -28, 42, -56, 86, -138, \ldots$$

These coefficients can be identified with McKay-Thompson series constructed out of trace of the element g corresponding to the $\mathbb{Z}_2$ involution of $K3$ embedded in M_{24}.

Cheng (2010), Gaberdiel, Hohenegger, Volpato (2010)

- Examine the new supersymmetric index in the $(0,1)$ sector

$$G^{(2)}(q) = -\frac{4}{3}\left[\frac{E_6 + 2\mathcal{E}_2(\tau)E_4}{\eta^{12}}\right].$$

$G^{(2)}$ is the generalization of

$$G^{(1)}(q) = -2\frac{E_6}{\eta^{12}}$$

which is the new supersymmetric index for $K3$ compactifications.

Decompose

$$G^{(2)}(q) = 8g_{h=\frac{1}{4},l=0}(\tau) + \sum_{n=0}^{\infty} A_n^{(2)} g_{h=n+\frac{1}{4},l=\frac{1}{2}}(\tau),$$

where

$$\begin{aligned}
g_{h=\frac{1}{4},l=0}(\tau) &= \frac{\theta_2^6}{\eta^6}\mathrm{ch}_{h=\frac{1}{4},l=0}(\tau,\frac{1}{2}) + q^{1/4}\frac{\theta_3^6}{\eta^6}\mathrm{ch}_{h=\frac{1}{4},l=0}(\tau,\frac{1+\tau}{2}) \\
&\quad -q^{1/4}\frac{\theta_4^6}{\eta^6}\mathrm{ch}_{h=\frac{1}{4},l=0}(\tau,\frac{\tau}{2}), \\
g_{h=\frac{1}{4},l=0}(\tau) &= \frac{\theta_2^6}{\eta^6}\mathrm{ch}_{h=\frac{1}{4},l=\frac{l}{2}}(\tau,\frac{1}{2}) + q^{1/4}\frac{\theta_3^6}{\eta^6}\mathrm{ch}_{h=\frac{1}{4},l=0}(\tau,\frac{1+\tau}{2}) \\
&\quad -q^{1/4}\frac{\theta_4^6}{\eta^6}\mathrm{ch}_{h=\frac{1}{4},l=0}(\tau,\frac{\tau}{2}).
\end{aligned}$$

The g 's are products of characters of D_6 and the $\mathcal{N}=4$ Virasoro characters.

- Substituting the expressions for g's we can solve for the coefficients $A_n^{(2)}$.

We have checked using Mathematica that the first 8 coefficients fall into the McKay-Thompson series for the $\mathbb{Z}_2$ involution embedded in M_{24}.

- The analysis can be repeated for other values of N.

The new supersymmetric index in the $(0,1)$ sector is given by

$$G^{(N)}(q) = \frac{-N}{\eta^{12}}\left[\frac{4}{N(N+1)}E_6 + \frac{4}{N+1}\mathcal{E}_N(\tau)E_4\right].$$

Decompose $G^{(N)}$ as

$$G^{(N)}(q) = \chi_N g_{h=\frac{1}{4},l=0}(\tau) + \sum_{n=0}^{\infty} A_n^{(N)} g_{h=n+\frac{1}{4},l=\frac{1}{2}}(\tau).$$

We can solve for the coefficients $A_n^{(N)}$

$$A_n^{(3)} = -2,\ 0,\ -6,\ 10,\ 0,\ -18,\ 20,\ 0,\ \ldots,$$
$$A_n^{(5)} = -2,\ 0,\ 2,\ 0,\ -6,\ 2,\ 0,\ 6,\ \ldots,$$
$$A_n^{(7)} = -2,\ -1,\ 0,\ 0,\ 4,\ 0,\ -2,\ 2,\ \ldots.$$

These are the coefficients of the McKay-Thompson series for the $\mathbb{Z}_N$ involution of $K3$ embedded in M_{24}

THRESHOLD CORRECTIONS

- Discuss the situation without the Wilson line and the $K3 \times T^2$ situation first.
- We then consider turn on the Wilson line and outline the modifications which arise.
- Finally we summarize the results for the CHL orbifolds.

• The moduli dependence of the one-loop running of the gauge group is given by

$$\Delta_G(T,U) = \int_{\mathcal{F}} \frac{d^2\tau}{\tau_2} \mathcal{B}_G,$$

$\mathcal{B}$ is a trace over the internal Hilbert space which is defined as

$$\mathcal{B}_G(\tau,\bar{\tau}) = \frac{1}{\eta^2} \mathrm{Tr}_R \left\{ F e^{i\pi F} q^{L_0 - \frac{c}{24}} \bar{q}^{\tilde{L}_0 - \frac{\tilde{c}}{24}} \left(Q^2(G) - \frac{1}{8\pi\tau_2} \right) \right\},$$

Q is the charge of the lattice vectors corresponding to the gauge group of interest.

$\mathcal{B}$ is closely related to the new supersymmetric index.

The term proportional to $1/8\pi\tau_2$ is the new supersymmetric index.

The new supersymmetric index for the $K3 \times T^2$ compactification is given by

$$\mathcal{Z}_{new}(q,\bar{q})^{(1)} = -8\Gamma_{2,2}\frac{1}{\eta^{24}}E_4(q)E_6(q)$$

$E_4(q)$ arises from the E_8 lattice,

$E_6(q)$ arises from the broken E_8' together with the $K3$.

• Consider the situation we are interested in the threshold corrections of the unbroken gauge group E_8.

The action $Q^2(E_8)$ is given by replacing

$$Q^2(E_8) \to -\frac{1}{8} q\partial_q E_4(q) = \frac{1}{24}(E_2 E_4 - E_6)$$

in the index. The coefficient $1/8$ can be fixed by modular invariance.

This results in the integrand

$$\mathcal{B}^{(1)}_{E_8}(\tau,\bar{\tau}) = -\frac{1}{3}\Gamma_{2,2}(q,\bar{q})\frac{1}{\eta^{24}}\left\{\left(E_2 - \frac{3}{\pi\tau_2}\right)E_4 E_6 - E_6 E_6\right\}.$$

Similarly we can compute the threshold corrections for the gauge group E_7.

$$\mathcal{B}^{(1)}_{E_7}(\tau,\bar{\tau}) = -\frac{1}{3}\Gamma_{2,2}(q,\bar{q})\frac{1}{\eta^{24}}\left\{\left(E_2 - \frac{3}{\pi\tau_2}\right)E_4E_6 - E_4^3\right\}.$$

• Consider the difference in the threshold integrands for the gauge groups

$$\begin{aligned}\mathcal{B}^{(1)}_{E_7} - \mathcal{B}^{(1)}_{E_8} &= \frac{1}{3\eta^{24}}\Gamma_{2,2}\left(E_4^3 - E_6^2\right), \\ &= 576\Gamma_{2,2}.\end{aligned}$$

To obtain the second line we have used the identity

$$E_4^3 - E_6^2 = 1728\eta^{24}.$$

The threshold integral simplifies drastically

$$\Delta_{E_7}^{(1)}(T,U) - \Delta_{E_8}^{(1)}(T,U) = 576 \int_{\mathcal{F}} \frac{d^2\tau}{\tau_2} \Gamma_{2,2}.$$

This integral can be done.
Dixon, Kaplunovsky, Louis (1991).

The result reduces to the product of the Dedekind η functions.

$$\Delta_{E_7}^{(1)}(T,U) - \Delta_{E_8}^{(1)}(T,U) = -48 \log(T_2^{12} U_2^{12} |\eta(T)\eta(U)|^{48}).$$

Wilson line turned on

• When the Wilson line is turned on, the lattice $\Gamma_{2,2}$ is enhanced to $\Gamma_{3,2}$.

The lattice sum is given by $\Gamma_{3,2}$ which is given by

$$\Gamma_{3,2} = \sum_{m_1, m_2, n_1, n_2, b} q^{\frac{p_L^2}{2}} \bar{q}^{\frac{p_R^2}{2}} .$$

$$\begin{aligned} \frac{p_R^2}{2} &= \frac{1}{4 \det \mathrm{Im} \Omega} \left| -m_1 U + m_2 + n_1 T + n_2 (TU - V^2) + bV \right|^2 , \\ \frac{p_L^2}{2} &= \frac{p_R^2}{2} + m_1 n_1 + m_2 n_2 + \frac{1}{4} b^2 . \end{aligned}$$

$$\Omega = \begin{pmatrix} U & V \\ V & T \end{pmatrix} .$$

The lattice sum over T^2 is characterized by the five charges (m_1, m_2, n_1, n_2, b).

• The new supersymmetric index with the Wilson line.

Re-write the lattice sum over E_8 in terms of a Jacobi form of index 1 given by

$$E_{4,1}(\tau, z) = \frac{1}{2}\left[\theta_2(\tau, z)^2\theta_2^6 + \theta_3(\tau, z)^2\theta_3^6 + \theta_4(\tau, z)^2\theta_4^6\right].$$

$E_{4,1}(\tau, 0) = E_4(q)$.

Essentially we have decomposed the E_8 lattice into D_6 and D_4 and

Introduced a chemical potential for a $U(1)$ in the D_4 sub-lattice.

Decompose this Jacobi form of index one into $SU(2)$ characters as follows

$$E_{4,1}(\tau, z) = E_{4,1}^{\mathrm{even}}(q)\theta_{\mathrm{even}}(\tau, z) + E_{4,1}^{\mathrm{odd}}(q)\theta_{\mathrm{odd}}(\tau, z).$$

$$\theta_{\mathrm{even}}(\tau, z) = \theta_3(2\tau, 2z), \qquad \theta_{\mathrm{odd}}(\tau, z) = \theta_2(2\tau, 2z).$$

$$\begin{aligned} E_{4,1}^{\mathrm{even}}(q) &= \frac{1}{2}\left(\theta_2(2\tau,0)\theta_2^6 + \theta_3(2\tau,0)\theta_3^6 + \theta_3(2\tau,0)\theta_4^6\right), \\ E_{4,1}^{\mathrm{odd}}(q) &= \frac{1}{2}\left(\theta_3(2\tau,0)\theta_2^6 + \theta_2(2\tau,0)\theta_3^6 - \theta_2(2\tau,0)\theta_4^6\right). \end{aligned}$$

The new supersymmetric index with the Wilson line turned on

$$\mathcal{Z}^{(1)}_{\rm new}(q,\bar{q}) \;=\; -8\,\frac{E_6}{\eta^{24}}\left(\sum_{\substack{m_1,m_2,n_1,n_2\in\mathbb{Z},\\ b\in 2\mathbb{Z}}} q^{\frac{p_L^2}{2}}\,\bar{q}^{\frac{p_R^2}{2}}\,E^{\rm even}_{4,1}(q)\right.$$

$$\left. +\sum_{\substack{m_1,m_2,n_1,n_2\in\mathbb{Z},\\ b\in 2\mathbb{Z}+1}} q^{\frac{p_L^2}{2}}\,\bar{q}^{\frac{p_R^2}{2}}\,E^{\rm odd}_{4,1}(q)\right).$$

p_L, p_R contain the Kähler, complex structure and the Wilson line moduli dependence of the T^2.

We have embedded the Wilson line in the group E_8

We can carry out a similar analysis for the group E_7.

Compactly we write the new supersymmetric index as

$$\mathcal{Z}^{(1)}_{\rm new}(q,\bar{q}) = -8\frac{E_6}{\eta^{24}} E_{4,1} \otimes \Gamma_{3,2}(q,\bar{q}).$$

The running of the group E_8 is now determined by the integrand

$$\mathcal{B}_G^{(1)}(\tau,\bar{\tau}) = -\frac{1}{3}\frac{1}{\eta^{24}}\left\{\left(E_2 - \frac{3}{\pi\tau_2}\right)E_{4,1}E_6 - E_{6,1}E_6\right\}\otimes\Gamma_{3,2}(q,\bar{q}).$$

The threshold integrand $\mathcal{B}_{G'}$ for group E_7 is given by

$$\mathcal{B}_{G'}^{(1)}(\tau,\bar{\tau}) = -\frac{1}{3}\frac{1}{\eta^{24}}\left\{\left(E_2 - \frac{3}{\pi\tau_2}\right)E_{4,1}E_6 - E_4^2E_{4,1}\right\}\otimes\Gamma_{3,2}(q,\bar{q}).$$

• Take the difference between threshold corrections corresponding to the two gauge groups.

$$\Delta^{(1)}_{G'}(T,U,V) - \Delta^{(1)}_{G}(T,U,V) = \int_{\mathcal{F}} \frac{d^2\tau}{\tau_2} \frac{1}{3\eta^{24}} \left(E_4^2 E_{4,1} - E_6 E_{6,1} \right) \otimes \Gamma_{3,2}(q,\bar{q}).$$

The combination of the Eisenstein series which occurs can be identified with the elliptic genus of $K3$ due to the following identities

$$\frac{1}{\eta^{24}} \left[E_4^2 E_{4,1}(\tau,z) - E_6 E_{6,1}(\tau,z) \right] = 72 Z_{K3}(\tau,z),$$

$$\frac{1}{\eta^{24}} \left[E_4^2 E_{4,1}^{\text{even,odd}} - E_6 E_{6,1}^{\text{even,odd}} \right] = 72 Z_{K3}^{\text{even,odd}}.$$

The integral can be performed and it results in

$$\Delta^{(1)}_{G'}(T,U,V) - \Delta^{(1)}_{G}(T,U,V) = -48\log\left[(\det\mathrm{Im}\Omega)^{10}|\Phi_{10}(T,U,V)|^2\right].$$

$\Phi_{10}(T,U,V)$ is the unique Siegel modular form of weight 10 under $Sp(2,\mathbb{Z})$ which is also known as the Igusa cusp form Stieberger (1999) .

• It is interesting that the difference in thresholds is in fact sensitive only the elliptic genus of $K3$.

• This property generalizes to the twisted elliptic genus for the CHL compactifications.

• The modular form Φ_{10} is associated with the elliptic genus of the symmetric product.

• It is also associated with the counting of the degeneracies of $1/4$ BPS states in heterotic string theories on T^6.

This theory has $\mathcal{N} = 4$ supersymmetry.

- The generalization for the $\mathbb{Z}_2$ CHL orbifold:

Without the Wilson line

$$\Delta^{(2)}_{E_7} - \Delta^{(2)}_{E_8} = -48 \log \left\{ T_2^8 U_2^8 |\eta(T)\eta(2T)|^{16} |\eta(U)\eta(2U)|^{16} \right\}.$$

With the Wilson line The integrands of the difference in thresholds is given by

$$\begin{aligned} \mathcal{B}^{(2)}_{G'} - \mathcal{B}^{(2)}_{G} &= 24 \left\{ \Gamma^{(0,0)}_{3,2} \otimes F^{(0,0)} + \Gamma^{(0,1)}_{3,2} \otimes F^{(0,1)} + \Gamma^{(1,0)}_{3,2} \otimes F^{(1,0)} \right. \\ &\qquad \left. + \Gamma^{(1,1)}_{3,2} \otimes F^{(1,1)} \right\}. \end{aligned}$$

Note that it the lattice sum folded with the twisted elliptic genus of $K3$.

Performing the modular integral results in

$$\Delta^{(2)}_{G'}(U,T,V) - \Delta^{(2)}_{G}(U,T,V) = -48 \log \left[(\det \mathrm{Im}\Omega)^6 |\Phi_6(U,T,V)|^2 \right].$$

The Siegel modular form, $\Phi_6(T,U,V)$, transforms as a weight 6 form under a subgroup of $Sp(2,\mathbb{Z})$.

• The modular form Φ_6 is also related to the partition function of $1/4$ BPS dyons in type II theory on the CHL orbifold of $K3$.

This theory has $\mathcal{N} = 4$ supersymmetry.

Let $\tilde{\Phi}_6$ be the generating function of dyons in this theory, then the modular form Φ_6 is related by the following $Sp(2, \mathbb{Z})$ transformation.

$$\Phi_6(U, T, V) = T^{-6}\tilde{\Phi}_6(U - \frac{V^2}{T}, -\frac{1}{T}, \frac{V}{T}).$$

• For the $\mathbb{Z}_N$ CHL orbifolds we obtain

$$\Delta_{G'}^{(N)}(U,T,V) - \Delta_{G}^{(N)}(U,T,V) = -48\log[(\det \mathrm{Im}\Omega)^k |\Phi_k(U,T,V)|^2].$$

Φ_k is the Siegel modular form of weight k transforming according to a subgroup of $Sp(2,\mathbb{Z})$.

CONCLUSIONS

• Introduced $\mathcal{N} = 2$ string theories constructed by compactifying heterotic string theories on CHL orbifolds of $K3$.

These generalize the well studied example of the heterotic string compactified on $K3 \times T^2$.

•Evaluated the new supersymmetric index for these compactifications .

•Studied the moduli dependence of one-loop corrections to the gauge couplings in the CHL orbifolds of $K3$.

• Observe that the difference in integrands of the gauge thresholds reduces to the twisted elliptic genus of $K3$ for the CHL orbifold.

Points to the fact that the difference in the thresholds is essentially sensitive only to a supersymmetric index of the internal CFT.

It will be interesting to prove this in general.

• Generalization: Consider compactifications in heterotic based on the new classes of twisted elliptic genera of $K3$ constructed by
Cheng (2010), Gaberdiel, Hohenegger, Volpato (2010)