SYK/Tensor models
and
2-dim Quantum Gravity

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Ref : arXiv : 1702.04266 V2
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String Theory and Quantum Gravity
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June 5, 2017

Sachdev-Ye-Kitaev Model

QM of N Majorana fermions

$\psi_i(t)\ ,\ i=1,\dots,N$ [Kitaev, M Idacuna Stanford]

$$H = \sum_{1\le i<j<k<l\le N} J_{ijkl}\,\psi_i\psi_j\psi_k\psi_l$$

J_{ijkl} is a gaussian random coupling

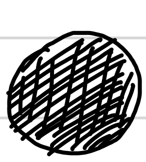
$$\langle J_{ijkl}\rangle = 0 \qquad \langle J_{ijkl}^2\rangle = \frac{6J^2}{N^3}$$

Model has 2 important properties:

1. $N\to\infty\ ,\ \beta J \gg 1$

Emergent reparametrization (almost)

2. $$\frac{\langle \psi_i(0)\psi_j(t)\psi_i(0)\psi_j(t)\rangle}{\langle\psi_i(0)\psi_i(0)\rangle\langle\psi_j(t)\psi_j(t)\rangle} \sim \frac{\beta J}{N}e^{\lambda_L t}$$



holes $\lambda_L = 2\pi/\beta\hbar$ Chaos bound

$t_s \sim \frac{\hbar}{kT}(S - \ln\beta J)$ Scrambling time

(large N + Strong Coupling)
Soluble model to study holographic duality and black hole physics

Averaging over disorder, the large N SD equations are expressed in terms of the bilocal fields:

$G(\tau_1,\tau_2)$ + $\Sigma(\tau_1,\tau_2)$ (euclidean time τ)

$$\frac{1}{G(\omega)} = -i\omega - \Sigma(\omega),$$

$$\Sigma(\tau_1,\tau_2) = J^2 G(\tau_1,\tau_2)$$

They are large N saddle points:

$$Z = \int \mathcal{D}G\,\mathcal{D}\Sigma\; e^{-NS}$$

$$S = -\tfrac{1}{2}\log\det(\partial_\tau - \Sigma) + \int d\tau_1 d\tau_2 \left(\Sigma(\tau_1,\tau_2)G(\tau_1,\tau_2) - \frac{J^2}{4} G(\tau_1,\tau_2)^4 \right)$$

Emergent reparametrization symmetry

$\omega << J$ (or $\beta J >> 1$ at finite temp)

action and SD eqns are invariant under:

$$G(\tau_1,\tau_2) \to \left[f'(\tau_1) f'(\tau_2)\right]^{\frac{1}{4}} G(f(\tau_1), f(\tau_2))$$

$$\Sigma(\tau_1,\tau_2) \to \left[f'(\tau_1) f'(\tau_2)\right]^{\frac{3}{4}} \Sigma(f(\tau_1), f(\tau_2))$$

Solution:

$$G_c(\tau) \sim \frac{\mathrm{Sgn}(\tau)}{|J\tau|^{1/4}}, \quad \Sigma_c(\tau) \sim J^2 G_c^3$$

finite temp. $\tau \to \tan\frac{\pi\tau}{\beta}$

$$G_c(\tau) \sim \mathrm{Sgn}(\tau)\left(\frac{\pi}{\beta J \sin\frac{\pi\tau}{\beta}}\right)^{\frac{1}{2}}$$

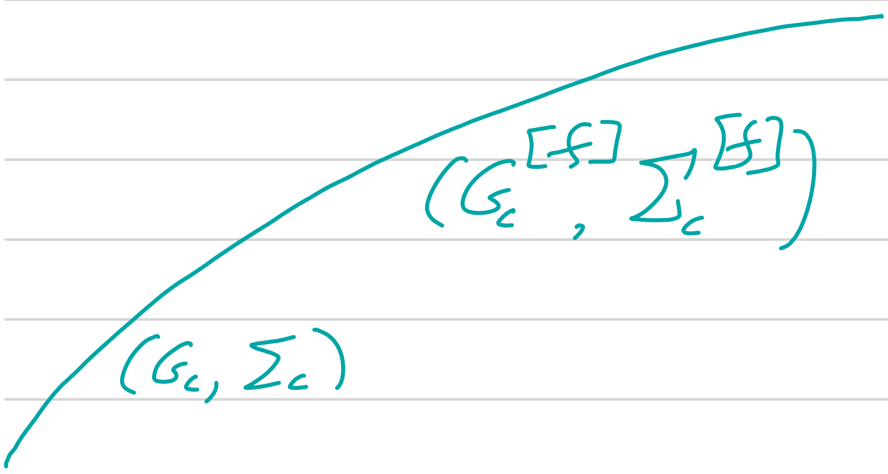
Diff(1) spontaneously broken to $SL(2,R)$

Spontaneous Symmetry breaking

$$\mathrm{Diff}(1) \to SL(2,R)$$

R^1 / S^1_β

$G_c^{[f]}(\tau)$, $\Sigma_c^{[f]}(\tau)$, $f(\tau) \in \mathrm{Diff}(1)/SL(2,R)$



$$S(G_c, \Sigma_c) = S(G_c^{[f]}, \Sigma_c^{[f]})$$

$$Z \sim \int_{\frac{\mathrm{Diff}(1)}{SL(2,R)}} d\mu(f)\, e^{-N S(G_c, \Sigma_c)} = \infty$$

[more careful analysis gives the same answer $Z \propto \mathrm{Vol}\left(\frac{\mathrm{Diff}(1)}{SL(2,R)}\right) = \infty$ as $\beta J \to \infty$]

There is a J independent spectrum of evenly $O(1)$ spaced dimensions $\{h_m\}$ $m = 1, 2, \dots \infty$ corresponding to $\delta G(\tau_1, \tau_2) = G(\tau_1, \tau_2) - G_c(\tau_1 - \tau_2)$

$$\sim \mathcal{O}_m = \sum_i \psi_i \partial_\tau^{2m+1} \psi_i, \quad m = 1, 2, \dots \infty$$

Treatment of the Diff(1) / SL(2,R) modes

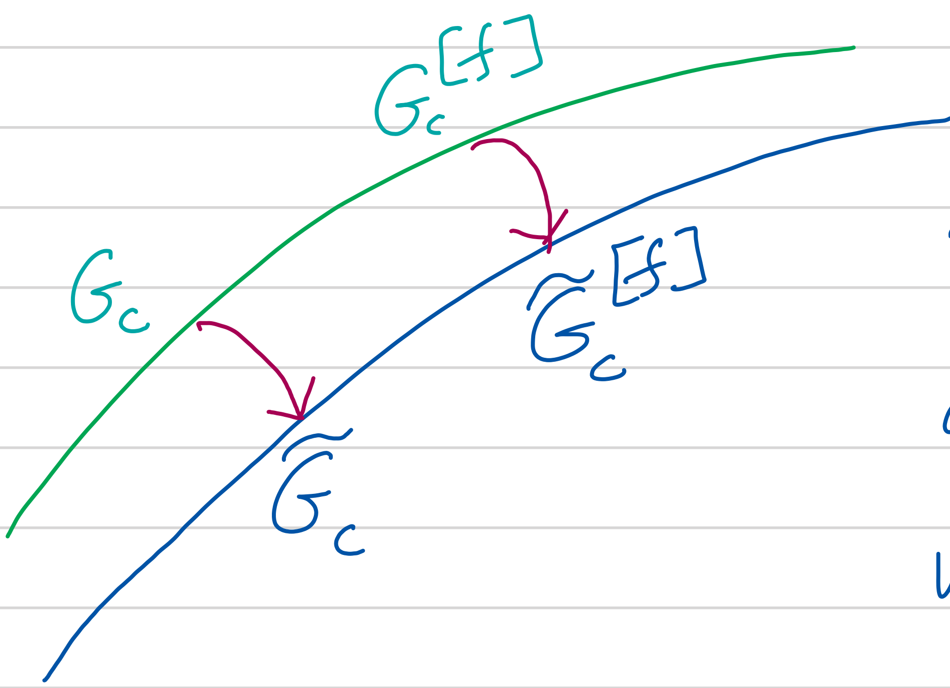
Solve SD eqns for large $J\tau$ / βJ :

$$\tilde{G}_c(\tau) = G_c(\tau)\left[1 - \frac{1}{2|J\tau|} + \cdots\right]$$

$$\Big\downarrow \; \tau \to \tan\frac{\pi\tau}{\beta}$$

$$\tilde{G}_c(\tau,\beta) = G_c(\tau,\beta)\left[1 - \frac{1}{2\beta J}\left(\frac{2+\pi-2\pi|\tau|/\beta}{\tan\frac{\pi|\tau|}{\beta}}\right)\cdots\right]$$

To explicitly break SL(2,R) symmetry



Perturbation theory around $\tilde{G}_c^{[f]}$.

well defined.

$\downarrow$

Effective action for low lying spectrum is the Schwarzian

The Schwarzian

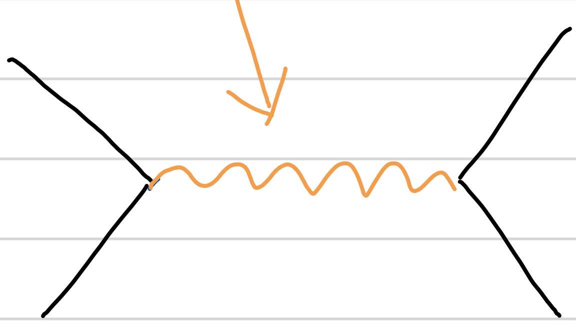
large N, large J

$$S_{Sch} \sim \frac{N}{J}\int_0^\beta d\tau \left[\left(\frac{f''}{f'}\right)^2 - \left(\frac{2\pi}{\beta}\right)^2 (f')^2\right]$$

(Kitaev, Maldacena Stanford, Jevicki, Yoon, Suzuki ...)

$$f(\tau) = \tau + \epsilon(\tau) \implies$$

$$\langle \epsilon(\tau)\epsilon(0)\rangle \sim \frac{\beta J}{N}\left(-\frac{(|\tau|-\pi)^2}{2} + (|u|-\pi)\sin|u|\right)$$



$$\sim \left(\frac{\beta J}{N}\right) e^{\frac{2\pi}{\hbar\beta} t}$$

real time

(dominant exchange in 4-point function).

Similar conclusions can be drawn for tensor models which do not have disorder.
(Witten, Gurau, Klebanov, Tarnopolsky)

SYK/Tensor models are models to study holography and black holes as indicated by the large N and strong coupling results. To resolve BH conundrums like the information paradox, behind the horizon issues etc one will need to solve these models at finite N, when the models have a discrete spectrum ...... then study the large N limit.

Holographic dual of SYK/Tensor models

1. Jackiw-Teitelboim dilaton gravity

(Maldacena, Stanford, Yang, Polchinski Almeheiri)

2. 2-dim. gravity with Polyakov action

(Mandal, Nayak, SRW 1702 04266 V2)

Motivation:

1 SYK model has emergent Diff(1) symmetry

$\xRightarrow{\text{broken}}$ $SL(2,R)$ [large N, strong coupling]

2. Quantization of the coadjoint orbit of

$Diff(S^1)/SL(2,R)$

$Diff(R^1)/SL(2,R)$

Witten
{Alexeev, Shatashvile
{Rodgers, Rai

$\Rightarrow$ Polyakov's 2-dim. gravity

Action $\propto \frac{1}{G_N}\int d^2x \, R \frac{1}{\Box} R \sqrt{g}$

3. Requirement of asymptotic AdS_2

geometry $\Longrightarrow$ $\mu \int d^2x \sqrt{g}$

Also $R\left(\frac{1}{\Box}R\right)^n$, $n>1$ not allowed.

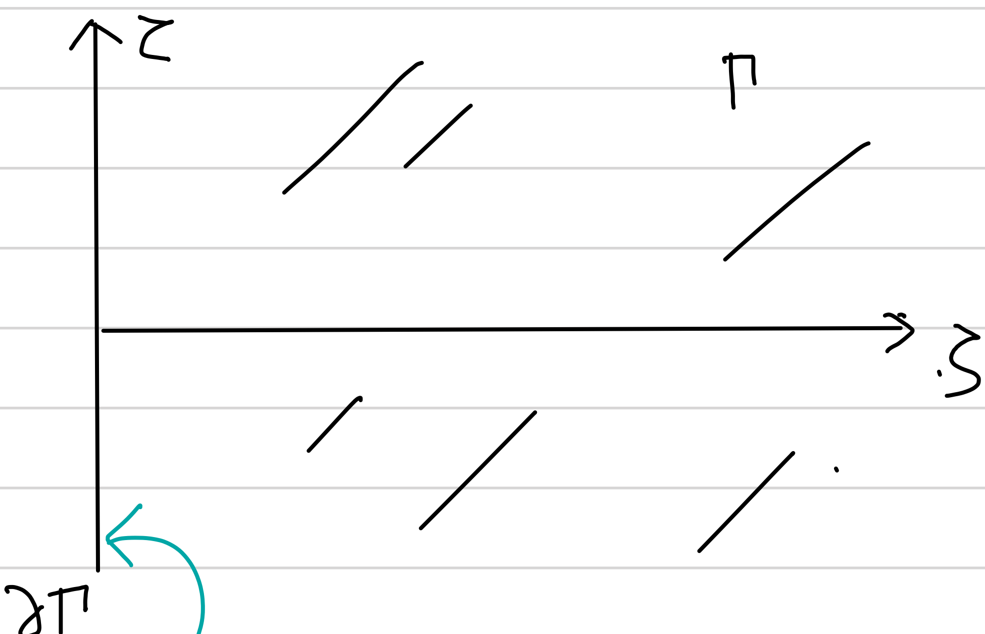
The model:

$$S[g] = \frac{1}{16\pi b^2}\int_\Gamma \sqrt{g}\left[R\frac{1}{\Box}R - 16\pi\mu\right]$$

extrinsic curvature on $\partial\Gamma$

$$\frac{1}{4\pi b^2}\int_{\partial\Gamma}\sqrt{\gamma}\left(K\frac{1}{\Box}R + K\frac{1}{\Box}K\right)$$

Domain Γ is the right half plane.



$$\frac{1}{\Box}R(x) = \int_\Gamma G(x,y)R(y)$$

SYK lives on $\partial\Gamma \simeq \mathbb{R}^1$

$b^2 = \frac{3}{2C}$ is the dimensionless Newton's constant

$-\mu < 0$ is the cosmological constant

Boundary terms are required so that eqns of motion follow from a variational principle.

Eqns of motion:

I $\quad R(x) = -8\pi\mu \quad < 0$

II $\quad \int_{\Gamma} d^2x \sqrt{g} \left(\nabla_\mu^{(\omega)} \nabla_\nu^{(\omega)} G(\omega, x) - \frac{1}{2} g_{\mu\nu}(\omega) \,\Box^{(\omega)} G(\omega, x) \right) R(x)$

$$= \frac{1}{2} \int_{\Gamma} d^2x \sqrt{g} \int_{\Gamma} d^2y \sqrt{g} \Big[\Big(\frac{\partial}{\partial \omega^\mu} G(\omega, x) \frac{\partial G(\omega, x)}{\partial \omega^\mu} - g_{\mu\nu}(\omega)\, g^{\alpha\beta}(\omega) \frac{\partial}{\partial \omega^\alpha} G(\omega, x) \frac{\partial}{\partial \omega^\beta} G(\omega, y) \Big) R(x) R(y) \Big]$$

If we use $\quad g_{\alpha\beta} = \hat{g}_{\alpha\beta} e^{\phi} \qquad \left(\begin{matrix} \hat{g} \to \hat{g} e^{\alpha} \\ \phi \to \phi - \alpha \end{matrix} \right)$

I $\quad -2 \hat{\Box} \phi + \hat{R} + 8\pi\mu e^{2\phi} = 0$

II $\quad \partial_\mu \phi \partial_\nu \phi - \nabla_\mu \nabla_\nu \phi - \frac{1}{2} \tilde{g}_{\mu\nu} \left(\partial^\alpha \phi \partial_\alpha \phi - 2 \hat{\Box} \phi \right)$

$$- \frac{1}{2} \tilde{g}_{\mu\nu} \left(4\pi\mu e^{2\phi} \right) = 0$$

follow from Liouville type action :

Liouville type action

$$S_L(\phi,\hat{g}) = -\frac{1}{4\pi b^2}\int_\Gamma \sqrt{\hat{g}}\left(\hat{g}^{\alpha\beta}\partial_\alpha\phi\,\partial_\beta\phi + \hat{R}\phi + 4\pi\mu\, e^{2\phi}\right)$$
$$+\frac{2}{4\pi b^2}\int_{\partial\Gamma}\sqrt{\hat{\gamma}}\;\hat{k}\,\phi$$
$$+\frac{1}{4\pi b^2}\int_{\partial\Gamma}\sqrt{\hat{\gamma}}\;\hat{n}^\mu\,\phi\,\partial_\mu\phi$$

Solutions

$$g_{\alpha\beta} = e^{2\phi}\,\hat{g}_{\alpha\beta}\ , \qquad d\hat{s}^2 = \hat{g}_{\alpha\beta}\,dx^\alpha dx^\beta$$
$$= \frac{d\zeta^2 + d\tau^2}{4\pi\mu\,\zeta^2}$$

Poincaré
right half plane: AdS_2
(a choice)

$$= \frac{dz\,d\bar{z}}{\pi\mu\,(z+\bar{z})^2}$$

I $\quad 2\,\hat{\Box}\phi = \hat{R} + 8\pi\mu\, e^{2\phi}\ , \qquad \hat{R} = -8\pi\mu$

$$\phi = \frac{1}{2}\log\left[(z+\bar{z})^2\,\frac{\partial g(z)\,\bar{\partial}\bar{g}(\bar{z})}{(g(z)+\bar{g}(\bar{z}))^2}\right]$$

Liouville

II $\quad \partial^2\phi - (\partial\phi(z,\bar z))^2 + 2\frac{\partial\phi(z,\bar z)}{(z+\bar z)} = 0$

$$\bar\partial^2\phi - (\bar\partial\phi(z,\bar z))^2 + 2\frac{\bar\partial\phi(z,\bar z)}{(z+\bar z)} = 0$$

Constraints

I + II $\Longrightarrow$ $\{g(z), z\} = 0$

$$\{\bar g(\bar z), \bar z\} = 0$$

$\{\ ,\ \}$ is the Schwarzian

$g(z) = \frac{az+ib}{icz+d}$, $\bar g(\bar z) = \overline{g(z)}$

$a, b, c, d \in \mathbb{C}$ $\quad ab + cd = 1$

If $g(z) + \bar g(\bar z)\Big|_{z+\bar z=0} = 0 \Rightarrow a, b, c, d \in \mathbb{R}^1$ hence $SL(2,\mathbb{R})$.

The rest of the 3 parameters lead to $g(z)$ which does not preserve the boundary $(z+\bar z = 0)$ of AdS_2 i.e. $g(z) + \bar g(\bar z)\Big|_{z+\bar z=0} \neq 0$

Small deformations of AdS_2

$a = 1 + \delta a\,, \quad b = \delta b\,, \quad c = \delta c\,, \quad d = 1 - \delta a$

$\Rightarrow$ solution

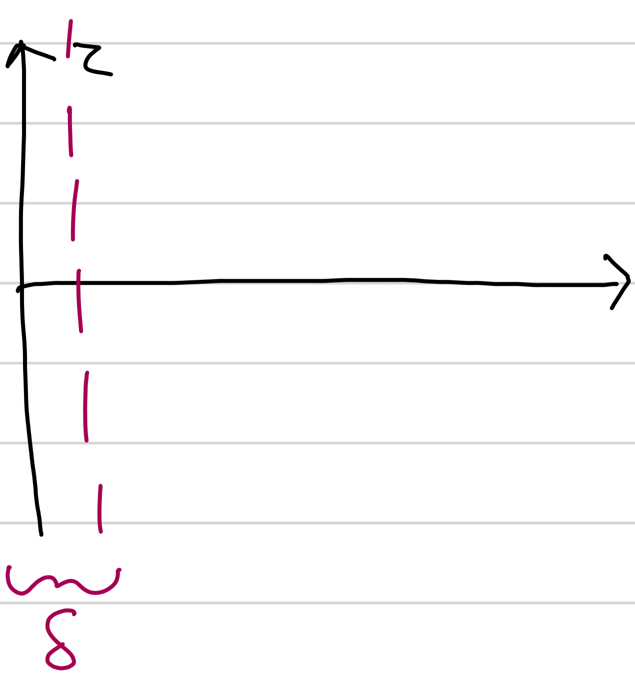
$$\phi = \frac{\delta g(\tau)}{\xi} + o(\delta a^2, \delta b^2, \delta c^2)$$

$$\delta g(\tau) = \mathrm{Im}(\delta b) + 2\,\mathrm{Im}(\delta a)\,\tau + \mathrm{Im}(\delta c)\,\tau^2$$

choose $\mathrm{Im}\,\delta a = \mathrm{Im}\,\delta c = 0$

$$\phi = \frac{\mathrm{Im}\,\delta b}{\xi} \equiv \frac{\delta g}{\xi} < 1$$

If $(\delta g) \lesssim \delta \to$ cutoff in AdS_2.



$$\boxed{\underbrace{ds^2 = \frac{1}{4\pi\mu}\,\frac{d\xi^2 + d\tau^2}{\xi^2}\left(1 + \frac{2\,\delta g}{\xi}\ldots\right)}_{NAdS_2}}$$

Asymptotically AdS_2 geometries

Fall off conditions!

1. $g_{\zeta\zeta} = \dfrac{1}{4\pi\mu\zeta^2}$, $g_{\zeta z} = O(\zeta^0)$, $g_{zz} = \dfrac{1}{4\pi\mu\zeta^2} + O(\zeta^0)$

2. Diff of AdS_2:

$$\delta g_{\alpha\beta} = \nabla_\alpha \epsilon_\beta + \nabla_\beta \epsilon_\alpha$$

$$= \begin{bmatrix} -\dfrac{\epsilon^\zeta - \zeta\partial_\zeta \epsilon^\zeta}{2\pi\mu\zeta^3} & \dfrac{\partial_z \epsilon^\zeta + \partial_\zeta \epsilon^z}{4\pi\mu\zeta^2} \\ \dfrac{\partial_z \epsilon^\zeta + \partial_\zeta \epsilon^z}{4\pi\mu\zeta^2} & -\dfrac{\epsilon^\zeta - \zeta\partial_z \epsilon^z}{2\pi\mu\zeta^3} \end{bmatrix}$$

3. Fix gauge Fefferman-Graham

$$\delta g_{\zeta\zeta} = 0 = \delta \dot{g}_{\zeta}$$

4. Solution for Killing vectors:

$$\epsilon^\zeta = \zeta\, \delta f(z), \quad \epsilon^z = \delta f(z) - \tfrac{1}{2}\zeta^2 \delta f''(z)$$

$\delta f(z)$ is arbitrary function $|\delta f| \ll 1$.

Diffs tangential to the boundary of AdS_2.

Finite transformations :

$$\tilde{z} = f(z) - \frac{2\zeta^2 f''(z) f'(z)^2}{4f'(z)^2 + \zeta^2 f''(z)^2} .$$

$$\tilde{\zeta} = \frac{4\zeta f'(z)^3}{4f'(z)^2 + \zeta^2 f''(z)^2}$$

can be obtained by restricting large diffs in AdS_3 to a plane $(\zeta, z, x=0)$.

$$\hat{ds}^2 = \frac{1}{4\pi\mu} \frac{\left(d\tilde{\zeta}^2 + d\tilde{z}^2\right)}{\tilde{\zeta}^2}$$

$$\Big\downarrow f(z)$$

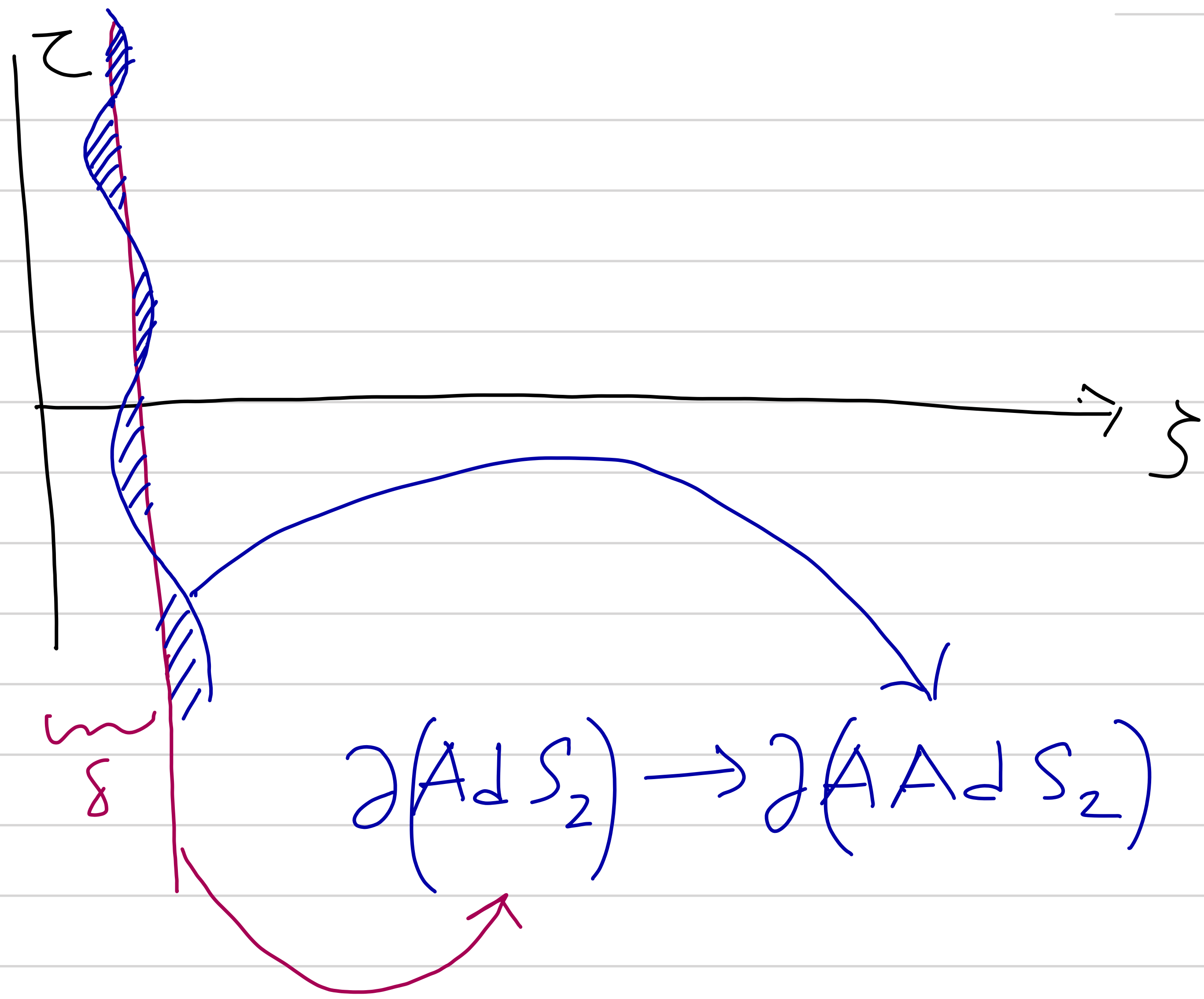
$\hat{g}[f]$

$$\boxed{\hat{ds}^2_f = \frac{1}{4\pi\mu\, \zeta^2}\left(d\zeta^2 + dz^2\left[1 - \zeta^2 \frac{\{f(z), z\}}{2}\right]^2\right) \qquad \{f, z\} = \frac{f'''}{f'} - \frac{3}{2}\left(\frac{f''}{f'}\right)^2 .}$$

Cut-off : $\tilde{\zeta} = \delta$

$$\Rightarrow \quad \zeta = \frac{2}{\delta} \frac{f'(z)}{f''(z)^2}\left[f'(z)^2 - \sqrt{f'^4 - \delta^2 f''(z)^2}\right]$$

$$\approx \delta\left(1 + \delta f'(z)\right) + \dots .$$



In summary in the limit $b^2 \to 0$ the classical geometries are entirely characterized by $\{f(\tau), f'(\tau) > 0\}$

$$\boxed{\begin{aligned}
ds^2 &= e^{2\phi}\, \hat{ds}^2 \\
\hat{ds}^2_f &= \frac{1}{4\pi\mu\, \zeta^2}\left(d\zeta^2 + d\tau^2\left[1 - \zeta^2\,\frac{\{f(\tau),\tau\}}{2}\right]^2\right) \\
\phi &= \frac{\delta g}{\zeta} + O(\delta a^2, \delta b^2, \delta c^2), \\
\delta g &= \mathrm{Im}(\delta b) + 2\,\cancel{\mathrm{Im}(\delta a)}\,\tau + \cancel{\mathrm{Im}(\delta c)}\,\tau^2
\end{aligned}}$$

The classical action:

Results:

1. $$S_{bulk}\left(\hat{g}^{[f]},\phi\right) - S_{bulk}\left(\hat{g}^{[f=\tau]},\phi\right)$$
$$= \frac{1}{4\pi b^2}\frac{1}{\delta}\left[\epsilon(\infty) - \epsilon(-\infty)\right]$$

where $f(\tau) = \tau + \epsilon(\tau)$ (note $f(\tau)=\tau$ is the identity diff)

choosing $\epsilon(\infty) = \epsilon(-\infty) \Rightarrow$

$$\boxed{\delta S_{bulk} = S_{bulk}\left(\hat{g}^{[f]},\phi\right) - S_{bulk}\left(\hat{g}^{[f=\tau]},\phi\right) = 0}$$

2. $$\delta S_{bd} = S_{bd}^{AAdS_2} - S_{bd}^{AdS_2}$$

$$\boxed{\delta S_{bd} = \frac{1}{2\pi b^2}\int d\tilde{\tau}\; \delta g(i\tilde{\tau})\left\{\tilde{f}(\tilde{\tau}),\tilde{\tau}\right\}}$$

2-dim gravity path integral

$$Z \sim \int \underbrace{\left[\frac{\mathcal{D}f(\tau)}{f'(\tau)}\right]'}_{\text{exclude integration over } SL(2,\mathbb{R}).} e^{-\frac{\delta g}{2\pi b^2}\int d\tau \{f(\tau),\tau\}}$$

At finite temperature using $\boxed{\tau = \tan\left(\frac{\pi\theta}{\beta}\right)}$

$$S_\beta = \frac{\delta g}{2\pi b^2}\int d\theta \left\{\frac{\beta}{2}\tan\left(\frac{\pi f(\theta)}{\beta}\right), \theta\right\}$$

Correspondence with SYK.

$$\delta g \sim \frac{1}{J} \quad , \quad b^2 \sim \frac{1}{N}$$

$$\ln Z = -\beta F = \underbrace{\frac{\delta g}{2b^2}\frac{1}{\beta}}_{\text{matches with SYK}} + (\text{constant})$$

Summary

- The 2-dim gravity dual which is naturally motivated by the infrared Virasoro symmetry correctly accounts for the hydrodynamic Schwarzian action.

- Around smooth AdS_2 + $AAdS_2$ geometries the only degree of freedom is the boundary diff. $\{f(z)\}$

- It would be interesting to understand the operators $\mathcal{O}_m = \sum_i \psi_i \partial_z^{2m+1} \psi_i$ $m>1$ in this dual framework.