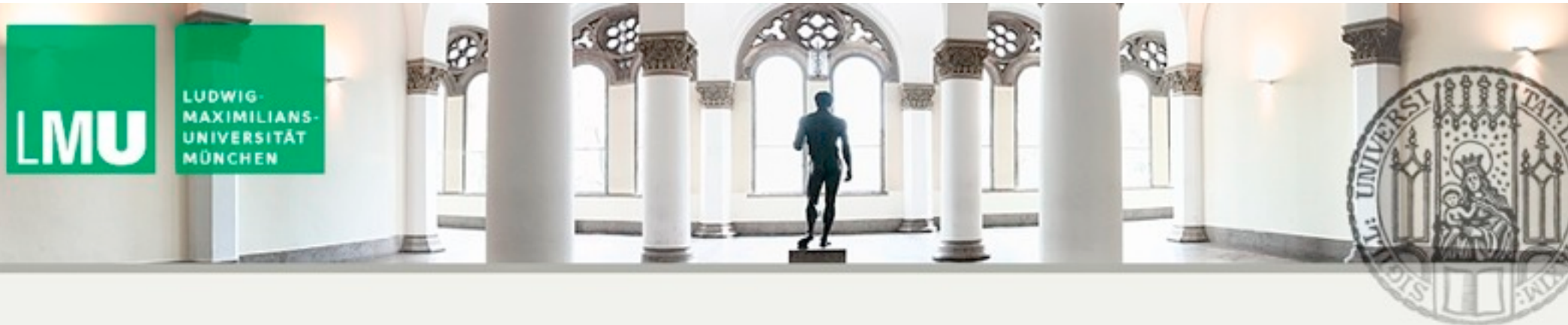# Non-Associativity in M-theory and the magnetic monopole/R-flux paradigm

DIETER LÜST (LMU, MPI)

String Theory and Quantum Gravity,   Monte Verita, July 4th, 2017

Non-Associativity in M-theory and the magnetic monopole/R-flux paradigm

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I. Bakas, D. L., arXiv:1309.3172

M. Günaydin, D.L., E. Malek, arXiv:1607.06474

D.L., E. Malek, R. Szabo, arXiv:1705.09639

String Theory and Quantum Gravity, Monte Verita, July 4th, 2017

This talk is dedicated to my friend Ioannis Bakas

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Outline:

II) Non-associative, octonionic algebra in M-theory from non-geometric R-flux - M2 brane phase space

II) Non-associative, octonionic algebra in M-theory from non-geometric Kaluza-Klein monopoles - M-wave phase space

III) Free M-theory phase space

II) Non-associative, octonionic algebra in M-theory from non-geometric R-flux - M2 brane phase space

Geometry in general depends on what kind of objects you test it.

Point particles in classical Einstein gravity „see" continuous Riemannian manifolds.

$$- \quad [x^i, x^j] = 0$$

Strings, membranes may see space-time in a different way.

We expect the emergence of a new kind of stringy geometry.

Consider three-dimensional string flux backgrounds:

Chain of three T-duality transformations:

(Hellerman, McGreevy, Williams (2002); C. Hull (2004); Shelton, Taylor, Wecht (2005); Dabholkar, Hull, 2005)

$$H_{ijk} \xrightarrow{T_i} f^i_{jk} \xrightarrow{T_j} Q^{ij}_k \xrightarrow{T_k} R^{ijk}, \quad (i,j,k = 1,\dots,3)$$

R-flux: locally non-geometric background
(similar to asymmetric orbifolds).

SO(3,3) Double field theory: $\quad R^{ijk} = 3\hat{\partial}^{[k}\beta^{ij]}$

bi-vector

Non-associative 6-dim. phase space of a probe string with 3 momenta and 3 coordinates in R-flux background:

$$\left[x^i, x^j\right] = i\frac{l_s^3}{\hbar}R^{ijk}p_k$$

$$\left[x^i, p^j\right] = i\hbar\delta^{ij}, \quad \left[p^i, p^j\right] = 0$$

$$\Longrightarrow \quad \left[x^i, x^j, x^k\right] \equiv \frac{1}{3}\left[\left[x^1, x^2\right], x^3\right] + \text{cycl. perm.} = l_s^3 R^{ijk}$$

R. Blumenhagen, E. Plauschinn, arXiv:1010.1263.

D.L., arXiv:1010.1361;

D. Mylonas, P. Schupp, R.Szabo, arXiv:1207.0926

I. Bakas, D.L., arXiv:1309.3172

This algebra can be derived from closed string CFT and applying duality symmetries.

R. Blumenhagen, A. Deser, D.L. , E. Plauschinn, F. Rennecke, arXiv:1106.0316

C. Condeescu, I. Florakis, D. L., arXiv:1202.6366

D. Andriot, M. Larfors, D.L. , P. Patalong:arXiv:1211.6437

C. Blair, arXiv:1405.2283

I. Bakas, D.L., arXiv:1505.04004

Two questions:

On the mathematical side:

How is the R-flux algebra related to other known non-associative algebras, in particular to the algebra of the octonions?

Our conjecture:

the answers to these two questions are closely related

On the physics side:

Can one lift the R-flux algebra of closed strings to M-theory?

R-flux algebra from octonions:

There exist four normed division algebras over $\mathbb{R}$:

$$\mathbb{R} \,, \mathbb{C} \,, \mathbb{Q} \,, \mathbb{O}$$

Division algebra of real octonions $\mathbb{O}$: non-commutative, non-associative
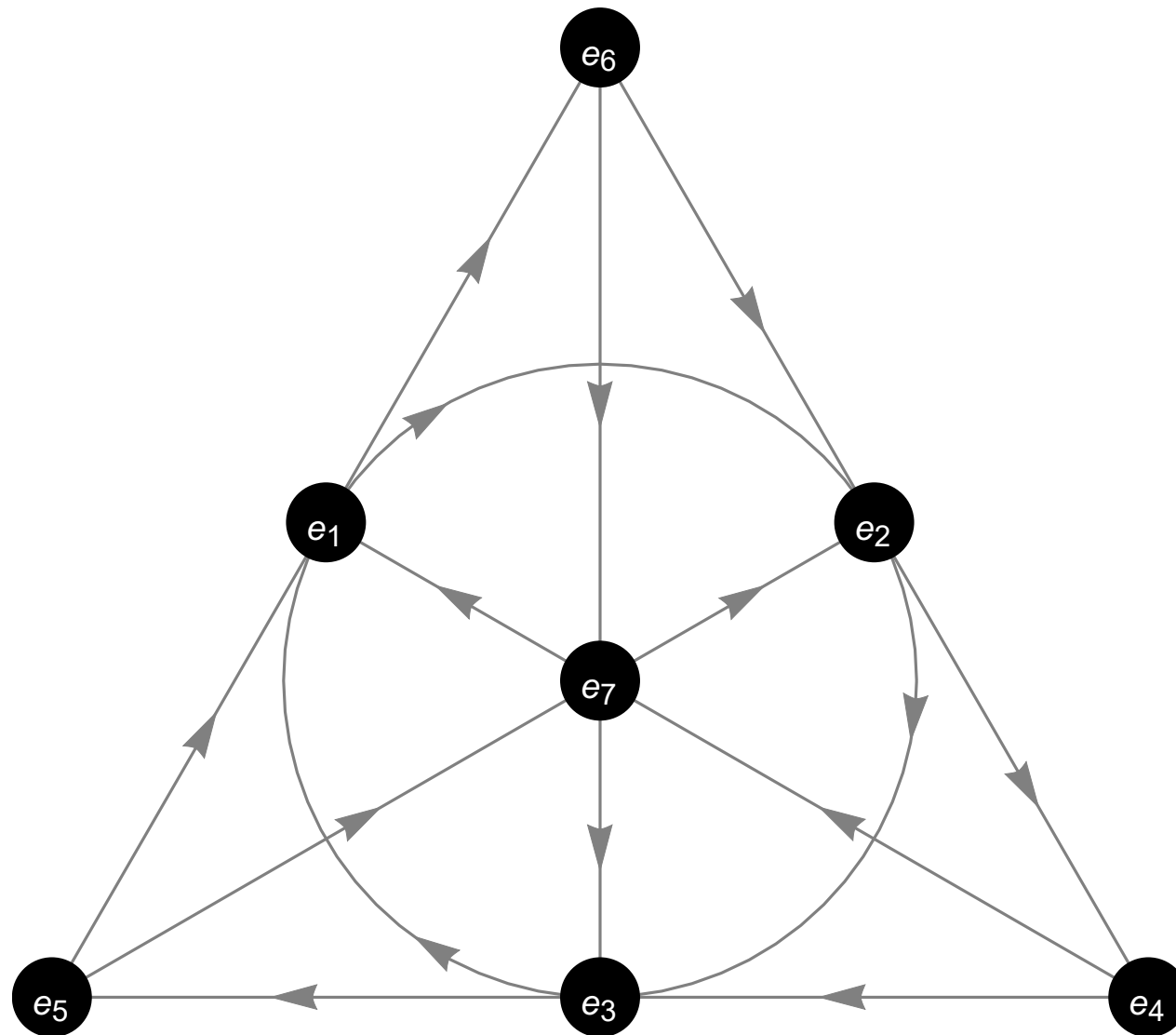
Besides the identity, there are seven imaginary units e_A

$$e_A e_B = \eta_{ABC}\, e_C \qquad (A = 1 \ldots, 7)$$

$$\eta_{ABC} = 1 \Longleftrightarrow (ABC) = (123),\, (516),\, (624),\, (435),\, (471),\, (572),\, (673)$$

This algebra is invariant under $G_2 \subset SO(7)$.

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Triangle of the seven octonions:

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Split indices: $\quad e_i \, , \quad e_{(i+3)} = f_i \, , \qquad$ for $i = 1, 2, 3$

$$\text{and} \quad e_7$$

Commutators:

$$[e_i, e_j] \quad = 2\epsilon_{ijk}e_k \, , \qquad [e_7, e_i] = 2f_i \, ,$$

$$[f_i, f_j] \quad = -2\epsilon_{ijk}e_k \, , \qquad [e_7, f_i] = -2e_i \, ,$$

$$[e_i, f_j] \qquad\qquad = 2\delta_{ij}e_7 - 2\epsilon_{ijk}f_k$$

3-brackets:

$$[e_i, e_j, f_k] \quad = 4\epsilon_{ijk}e_7 - 8\delta_{k[i}f_{j]} \, ,$$

$$[e_i, f_j, f_k] \qquad = -8\delta_{i[j}e_{k]} \, ,$$

$$[f_i, f_j, f_k] \qquad = -4\epsilon_{ijk}e_7 \, ,$$

$$[e_i, e_j, e_7] \qquad = -4\epsilon_{ijk}f_k \, ,$$

$$[e_i, f_j, e_7] \qquad = -4\epsilon_{ijk}e_k \, ,$$

$$[f_i, f_j, e_7] \qquad = 4\epsilon_{ijk}f_k$$

Associator $\quad [X, Y, Z] \equiv (XY)Z - X(YZ)$

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Now we rename the octonions in the following way:

$$p_i = -i\lambda \frac{1}{2} e_i \,, \qquad x^i = i\lambda^{1/2} \frac{\sqrt{R}}{2} f_i \,, \quad I = i\lambda^{3/2} \frac{\sqrt{R}}{2} e_7$$

Contraction of octonionic algebra: $\quad \lambda \to 0$

$$[f_i, f_j] = -2\epsilon_{ijk} e_k \quad \implies \quad [x^i, x^j] = iR\epsilon^{ijk} p_k$$

$$[e_i, e_j] = 2\epsilon_{ijk} e_k \quad \implies \quad [p_i, p_j] = 0$$

$$[f_i, e_j] = -\delta^i_j e_7 + \epsilon^i{}_{jk} f_k \quad \implies \quad [x^i, p_j] = i\delta^i_j I$$

$$[x_i, I] = 0 = [p_i, I]$$

$$[f_i, f_j, f_k] = -4\epsilon_{ijk} e_7 \quad \implies \quad [x^i, x^j, x^k] = R\epsilon^{ijk} I$$

Agrees with non-associative R-flux algebra !

What is the role of un-contracted algebra?

Lift of string R-flux algebra to non-geometric M-theory background:

- e^7 additional M-theory coordinate x^4

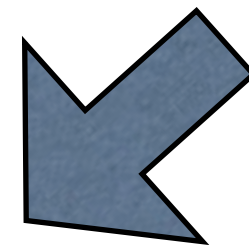 $\Rightarrow$ Four coordinates: f_1, f_2, f_3, e_7

- but no additional momentum.

  $\Rightarrow$ Three momenta: e_1, e_2, e_3

M-theory uplift of R-flux background:

Consider IIA string: the duality chain splits into two possible T-dualities:

$$H_{ijk} \xrightarrow{T_{ij}} Q^{ij}_k \, , \qquad f^i_{jk} \xrightarrow{T_{jk}} R^{ijk}$$

Uplift to M-theory: add additional circle $S^1_{x^4}$

- 3-dim IIA flux background $\Leftrightarrow$ 4-dim M-theory flux background

- two T-dualities $\Leftrightarrow$ 3 U-dualities

(Need third duality along the M-theory circle to ensure right dilaton shift.)

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Using SL(5) exceptional field theory one can obtain the R-flux in M-theory.

C. Blair, E. Malek, arXiv:1412.0635.

Non-geometric R-flux background in M-theory, which is dual to twisted torus:

R-flux:
$$R^{\alpha,\beta\gamma\delta\rho} = 4\hat{\partial}^{\alpha[\beta}\Omega^{\gamma\delta\rho]}$$

tri-vector

This is supposed to be still a consistent M-theory background, which can be probed by M2 branes.

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Four coordinates: x^1, x^2, x^3, x^4

What are the possible conjugate momenta (or windings)?

- Consider cohomology of twisted torus:

$$H^1(\tilde{T}^3 \times S^1, \mathbb{R}) = \mathbb{R}^3$$

- Alternatively consider Freed-Witten anomaly:

R-Flux with probe momentum p_4 along x^4
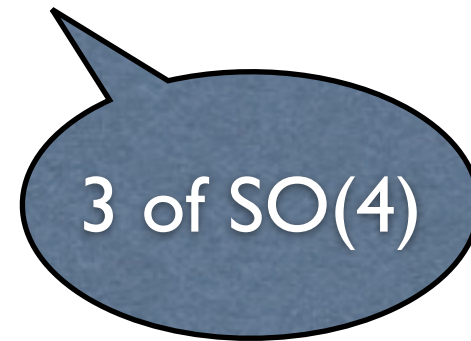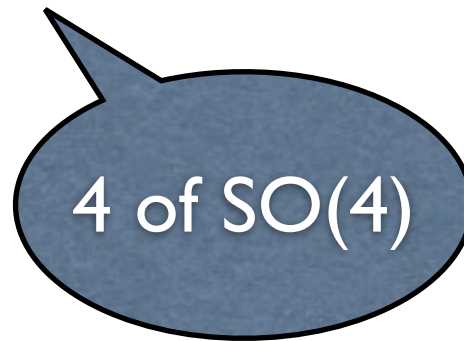
$\Updownarrow$

R-Flux with D0 branes.

Dualize to IIB: H-flux with D3-branes

This is forbidden by the Freed-Witten anomaly.

$\Rightarrow$ No momentum modes along the x^4 direction !

So we see that the phase space space of a M2-brane in the R-flux background in M-theory is seven-dimensional:

$$x^1, x^2, x^3, x^4 \; ; \; p_1, p_2, p_3$$

4 of SO(4)

3 of SO(4)

Missing momentum constraint in covariant terms:

$$F \equiv p_\alpha R^{\alpha, \beta\gamma\delta\rho} = 0$$

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Non-associative phase space algebra of M2-brane probes in R-flux background:

$$[x^i, x^j] = \frac{\ell_s^3}{\hbar} R^{4,ijk4} p_k , \qquad [x^4, x^i] = \frac{\lambda \ell_s^3}{\hbar} R^{4,1234} p^i ,$$

$$[x^i, p_j] = \hbar \delta_j^i x^4 + \hbar \lambda \varepsilon^i{}_{jk} x^k , \qquad [p_i, x^4] = \hbar \lambda^2 x_i ,$$

$$[p_i, p_j] = -\hbar \lambda \varepsilon_{ijk} p^k ,$$

$$[\![x^i, x^j, x^k]\!] = -\ell_s^3 R^{4,ijk4} x^4 ,$$

$$[\![x^i, x^j, x^4]\!] = \lambda^2 \ell_s^3 R^{4,ijk4} x_k ,$$

$$[\![p_i, x^j, x^k]\!] = -\lambda \ell_s^3 R^{4,1234} \left(\delta_i^j p^k - \delta_i^k p^j \right) ,$$

$$[\![p_i, x^j, x^4]\!] = -\lambda^2 \ell_s^3 R^{4,ijk4} p_k ,$$

$$[\![p_i, p_j, x^k]\!] = \hbar^2 \lambda^2 \varepsilon_{ij}{}^k x^4 + \hbar^2 \lambda \left(\delta_j^k x_i - \delta_i^k x_j \right) ,$$

$$[\![p_i, p_j, x^4]\!] = -\hbar^2 \lambda^3 \varepsilon_{ijk} x^k ,$$

$$[\![p_i, p_j, p_k]\!] = 0 ,$$

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Corresponding octonionic triangle:
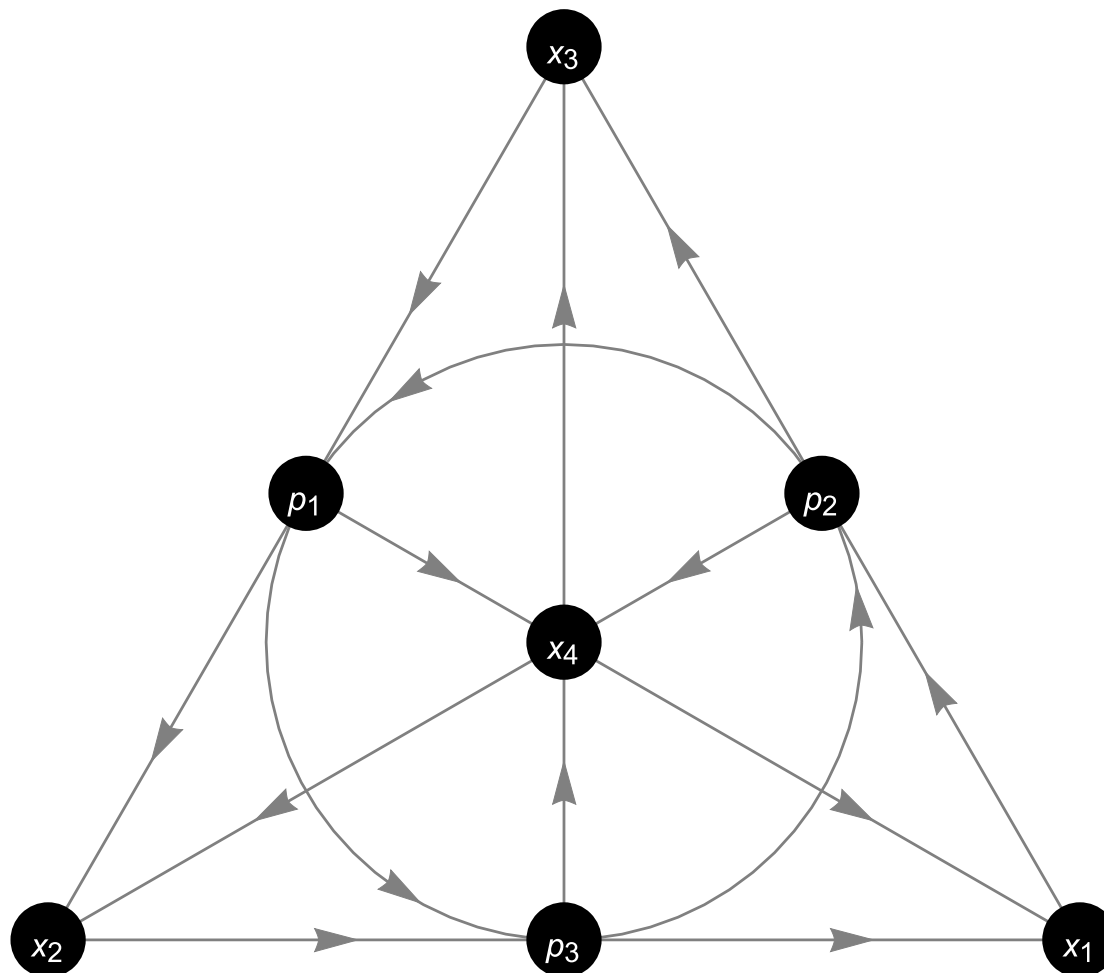
Natural identification of contraction parameter λ :

$$\lambda = \frac{\ell_{\mathrm{P}}}{\hbar}$$

Standard identification of M-theory parameters:

$$\ell_s^2 = \frac{\ell_{\mathrm{P}}^3}{r_{11}} \;, \qquad g_s = \left(\frac{r_{11}}{\ell_{\mathrm{P}}} \right)^{3/2}$$

(E. Witten, 1995)

Reduction to string theory

$$\ell_{\mathrm{P}} \sim r_{11}^{1/3} \to 0 \,, \quad g_s \sim r_{11} \to 0$$

Corresponds indeed to $\quad \lambda \to 0 \,,$

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II) Non-associative, octonionic algebra in M-theory from non-geometric Kaluza-Klein monopoles - M-wave phase space

Now we consider the phase space of an electron moving in the field of a magnetic monopole:

$$\left[x^i, x^j\right] = 0 \; , \qquad \left[x^i, p_j\right] = \hbar\, \delta^i_j \; , \qquad \left[p_i, p_j\right] = \frac{e\,\hbar}{c}\, \varepsilon_{ijk} B^k \; ,$$

$$\llbracket p_i, p_j, p_k \rrbracket = -\frac{e\,\hbar^2}{c}\, \varepsilon_{ijk}\, \nabla \cdot \vec{B}$$

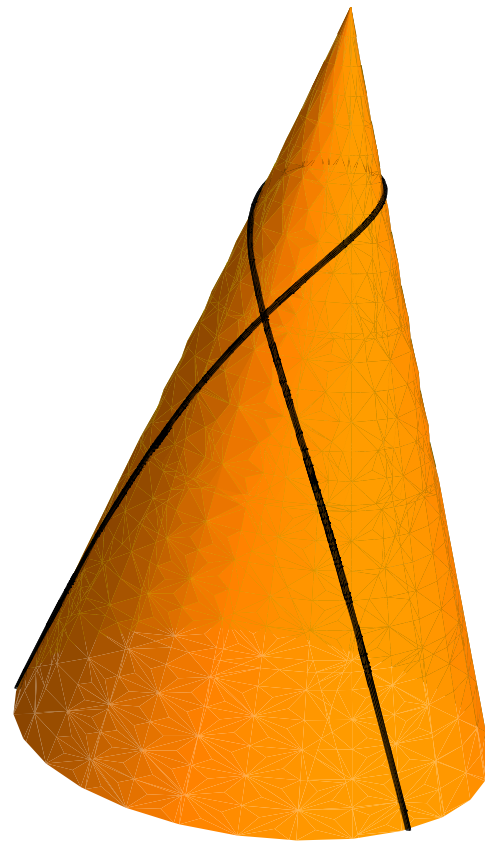p_i : gauge invariant momenta.

R. Jackiw (1985); B. Grossman (1985); Y. Wu, A. Zee (1985); J. Mickelsson (1986), M. Günaydin, B. Zumino (1985)

B^k : magnetic field of magnetic monopole

Donnerstag, 6. Juli 17

Isolated magnetic charges: the electron avoids the non-associativity by excising these points on position space - electron never reaches the magnetic monopole:

I. Bakas, D.L. (2013)

However we like to consider a constant smeared magnetic charge density ρ : magnetic charge is uniformly distributed over space.

$\Rightarrow$ If you now wanted remove the magnetic sources from position space you would end up with empty space.

(Signal of local non-geometry just like R-flux.)

This is described by a constant magnetic charge gerbe that is realized as a family of Dirac monopole gerbes.

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Phase space algebra of electron in constant magnetic charge density ρ :

$$[x^i, x^j] = 0 \ ,$$

$$[x^i, p_j] = \hbar\, \delta^i_j \ , \qquad\qquad [\![p_i, p_j, p_k]\!] = -\frac{\hbar^2\, e}{c}\, \rho\, \varepsilon_{ijk}$$

$$[p_i, p_j] = \frac{\hbar\, e}{c}\, \rho\, \varepsilon_{ijk}\, x^k \ ,$$

This algebra is isomorphic to R-flux algebra via the following (canonical) transformation:

$$x^i_R \longleftrightarrow p_{M\,i} \ , \quad p_{R\,i} \longleftrightarrow -x^i_M \ ,$$

$$\ell^3_s \longleftrightarrow \frac{e\,\hbar^2}{c} \ , \quad R \longleftrightarrow \rho$$

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M-theory up-lift of magnetic monopole algebra:

Conjecture: the magnetic charge algebra in M-theory is again provided by the non-associative algebra of the seven octonions.

But now there will be four momenta and three positions!

(i) What is the M-theory background which realizes this picture?

(ii) What are the relevant M-theory probes that have this octonionic algebra as phase space?

Donnerstag, 6. Juli 17

(i) KK monopole as M-theory background:

$$ds^2 = ds^2_{M_7} + U\, d\vec{y} \cdot d\vec{y} + U^{-1} \left(dz + \vec{A} \cdot d\vec{y} \right)^2 ,$$

$$A = \vec{A}(\vec{y}) \cdot d\vec{y} \qquad\qquad (\vec{y}, z) \in R^3 \times S^1$$

$$F = *_3\, dU \qquad\qquad A : \text{U(1) gauge connection}$$

We need a smeared KK monopole with constant magnetic charge density.

$\Rightarrow$ No local, geometric expression for metric and A:

A in non-local, smeared form:

$$A_i(\vec{y}) = \frac{1}{4\pi} \int A_i^{\mathrm{D}}(\vec{y} - \vec{y}')\, d^3\vec{y}' = \frac{N}{4\pi}\, \varepsilon_{ijk} \int \frac{(y_j - y_j')}{|\vec{y} - \vec{y}'| \left((y_k - y_k') + |\vec{y} - \vec{y}'| \right)}\, d^3\vec{y}' ,$$

(ii) M-wave as M-theory probe:

We need a probe that is electrically charged under the U(1) gauge field.

The electric dual to the KK monopole NUT charge is a graviton momentum mode $\Rightarrow$ M-wave

This M-wave travels along the x^4 direction.

$\Rightarrow$ No well-defined, local position with respect to x^4.

(IIA: D6 $\leftrightarrow$ D0 , $p_4 = \dfrac{\hbar e}{r_{11}}$)

Seven-dimensional phase space

$$(x^1, x^2, x^3; p_1, p_2, p_3, p_4)$$

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So we obtain as phase space algebra of a M-wave in the non-geometric KK-monopole background:

$$[x^i, x^j] = -\hbar\,\lambda\,\varepsilon^{ijk}\,x_k \ ,$$

$$[p_i, x^j] = \hbar\,\delta_i^j\,p_4 + \hbar\,\lambda\,\varepsilon_i{}^{jk}\,p_k \ , \qquad [x^i, p_4] = \hbar\,\lambda^2\,p^i \ ,$$

$$[p_i, p_j] = \frac{\hbar\,e}{c}\,\rho\,\varepsilon_{ijk}\,x^k \ , \qquad [p_4, p_i] = \frac{\hbar\,\lambda\,e}{c}\,\rho\,x_i \ ,$$

$$[x^i, x^j, x^k] = -2\,\hbar^2\lambda\,\varepsilon^{ijk}\,x^4 \ , \qquad [x^i, x^j, x^4] = \frac{\hbar^2\,\lambda^2}{2}\,\varepsilon^{ijk}\,x_k \ ,$$

$$[p^i, x^j, x^k] = -\frac{\hbar^2\,\lambda^2}{2}\,\varepsilon^{ijk}\,p_4 - \frac{\hbar^2\,\lambda}{2}\left(\delta^{ij}\,p^k - \delta^{ik}\,p^j\right) \ ,$$

$$[p_i, x^j, x^4] = -\frac{\hbar^2\,\lambda}{2}\,\delta_i^j\,p_4 - \frac{\hbar^2\,\lambda^2}{2}\,\varepsilon^{ijk}\,p_k \ ,$$

$$[p_i, p_j, x^k] = \frac{\hbar^2\,\lambda^2\,e}{2c}\,\rho\,\varepsilon_{ij}{}^k\,x^4 + \frac{\hbar^2\,\lambda\,e}{2c}\,\rho\left(\delta_j^k\,x_i - \delta_i^k\,x_j\right) \ ,$$

$$[p_i, p_j, x^4] = \frac{\hbar^2\,e}{2c}\,\rho\,\varepsilon_{ijk}\,x^k \ , \qquad [p_i, p_j, p_k] = -\frac{\hbar^2\,e}{2c}\,\rho\,\varepsilon_{ijk}\,p_4 \ ,$$
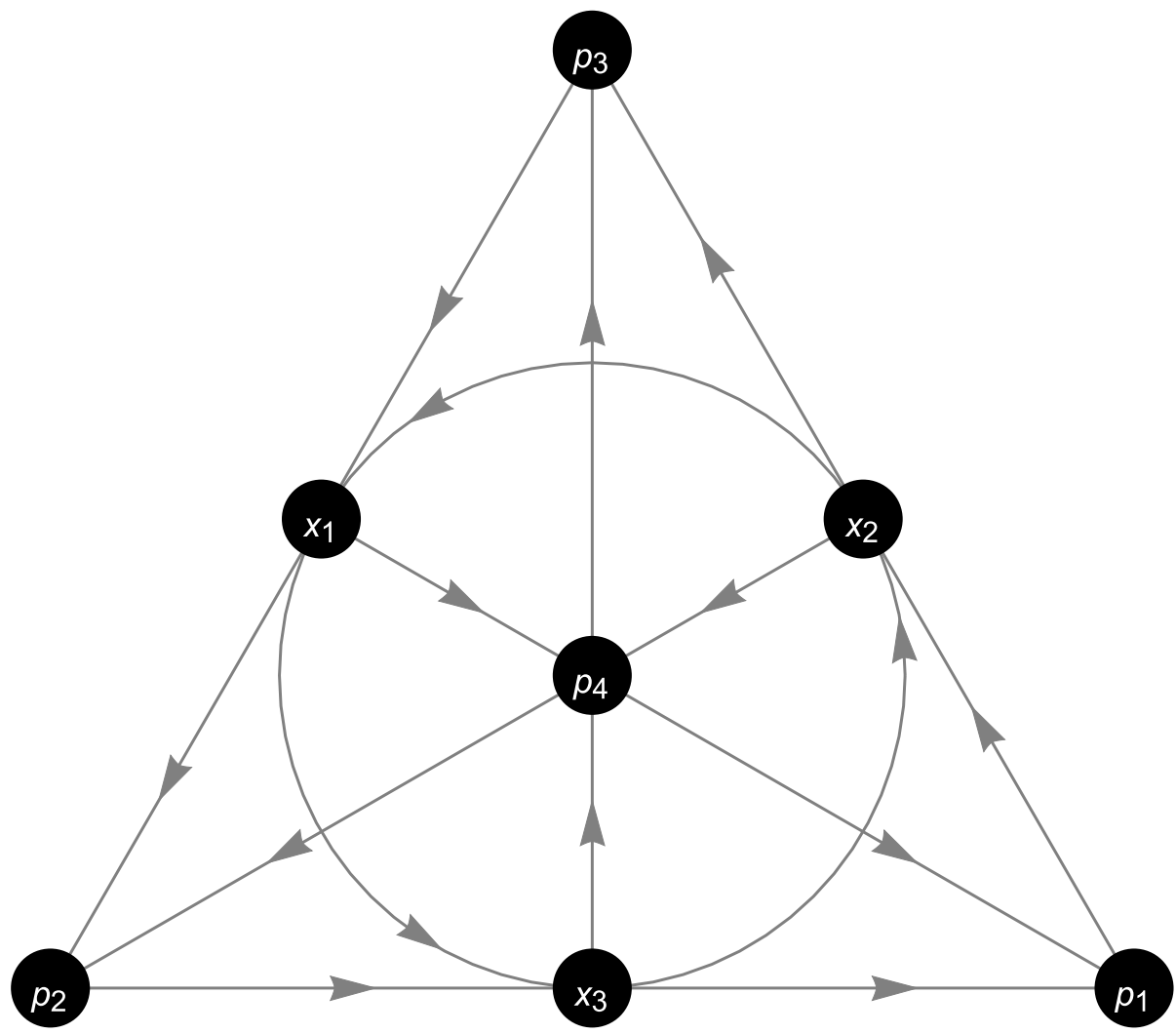
$$[p_4, x^i, x^j] = -\frac{\hbar^2\,\lambda^3}{2}\,\varepsilon^{ijk}\,p_k \ , \qquad [p_4, x^i, x^4] = -\frac{\hbar^2\,\lambda^3}{2}\,p^i \ ,$$

$$[p_4, p_i, x^j] = -\frac{\hbar^2\,\lambda^2\,e}{2c}\,\rho\,\delta_i^j\,x^4 - \frac{\hbar^2\,\lambda^2\,e}{2c}\,\rho\,\varepsilon_i{}^{jk}\,x_k \ ,$$

$$[p_4, p_i, x^4] = \frac{\hbar^2\,\lambda^2\,e}{2c}\,\rho\,x_i \ , \qquad [p_4, p_i, p_j] = \frac{\hbar^2\,\lambda^2\,e}{2c}\,\rho\,\varepsilon_{ijk}\,p^k \ .$$

$$\lambda \to 0\ (l_P, r_{11} \to 0): \quad \text{magnetic charge algebra}$$

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Corresponding octonionic triangle:

Magnetic monopoles and quantum gravity:

Restrict momenta to be on the unit sphere:

Then we obtain in the limit of $\rho \to 0$:

$$[x^i, x^j] = -\hbar \lambda \, \varepsilon^{ijk} \, x_k \ ,$$

$$[x^i, q_j] = \hbar \sqrt{1 - \lambda^2 \, |\vec{q}\,|^2} \ \delta^i_j + \hbar \lambda \, \varepsilon^i_{\ jk} \, q^k \ ,$$

$$[q_i, q_j] = 0 \ .$$

L. Freidel, E. Livine (2005)

These are the commutation relations of Ponzano-Regge spin foam model of three-dimensional quantum gravity.

Octonionic magnetic algebra can be viewed as algebra of monopoles in quantum gravity.

III) Free M-theory phase space

Assume no missing momentum/coordinate condition.

$p_4\left(x^4\right)$ corresponds to the 8th. (identity) octonion e_8 .

3-algebra: $\quad \left[\Xi_{\hat{A}}, \Xi_{\hat{B}}, \Xi_{\hat{C}}\right] = -2\,\hbar^2\,\phi_{\hat{A}\hat{B}\hat{C}\hat{D}}\,\Xi_{\hat{D}}$

$$\hat{A}, \hat{B}, \hat{C} = 1, \dots, 8$$

This algebra is invariant under $Spin(7) \subset SO(8)$.

Failure of defining Nambu-Poisson bracket:

$$\left[\!\left[\Xi_{\hat{A}}, \Xi_{\hat{B}}, \Xi_{\hat{C}}, \Xi_{\hat{D}}, \Xi_{\hat{E}}\right]\!\right] := \tfrac{1}{12}\left(\left[\left[\Xi_{\hat{A}}, \Xi_{\hat{B}}, \Xi_{\hat{C}}\right], \Xi_{\hat{E}}, \Xi_{\hat{D}}\right] + \left[\left[\Xi_{\hat{A}}, \Xi_{\hat{B}}, \Xi_{\hat{D}}\right], \Xi_{\hat{C}}, \Xi_{\hat{E}}\right]\right.$$

$$\left. + \left[\left[\Xi_{\hat{A}}, \Xi_{\hat{B}}, \Xi_{\hat{E}}\right], \Xi_{\hat{C}}, \Xi_{\hat{D}}\right] + \left[\left[\Xi_{\hat{C}}, \Xi_{\hat{D}}, \Xi_{\hat{E}}\right], \Xi_{\hat{A}}, \Xi_{\hat{B}}\right]\right)$$

$$= \hbar^4\left(\delta_{\hat{A}\hat{C}}\,\phi_{\hat{D}\hat{E}\hat{B}\hat{F}} + \delta_{\hat{A}\hat{D}}\,\phi_{\hat{E}\hat{C}\hat{B}\hat{F}} + \delta_{\hat{A}\hat{E}}\,\phi_{\hat{C}\hat{D}\hat{B}\hat{F}}\right.$$

$$\left. - \delta_{\hat{B}\hat{C}}\,\phi_{\hat{D}\hat{E}\hat{A}\hat{F}} - \delta_{\hat{B}\hat{D}}\,\phi_{\hat{E}\hat{C}\hat{A}\hat{F}} - \delta_{\hat{B}\hat{E}}\,\phi_{\hat{C}\hat{D}\hat{A}\hat{F}}\right)\Xi^{\hat{F}}$$

$$- \hbar^4\left(\phi_{\hat{B}\hat{C}\hat{D}\hat{E}}\,\Xi_{\hat{A}} - \phi_{\hat{A}\hat{C}\hat{D}\hat{E}}\,\Xi_{\hat{B}}\right) .$$

Any constraint $f(\Xi) = 0$ breaks

$$Spin(7) \longrightarrow G_2 \qquad \mathbf{8} = 7 \oplus 1$$

$p_4 = 0 \implies$ M-theory R-flux algebra.

$x^4 = 0 \implies$ M-theory monopole algebra.

The exchange of coordinates and momenta corresponds to a Spin(7) transformation - exchange of two SU(2)'s :

$$Spin(7) \supseteq SU(2)^3 \longrightarrow SU(2)^2 \subseteq G_2$$

$$\mathbf{8}\big|_{SU(2)^3} = (2,1,2) \oplus (1,2,2) \longrightarrow \mathbf{8}\big|_{SU(2)^2} = (2,2) \oplus (1,1) \oplus (1,3)$$

R-flux: x^α p_α x^α p_i

Any constraint $f(\Xi) = 0$ breaks

$$Spin(7) \longrightarrow G_2 \qquad \mathbf{8} = \mathbf{7} \oplus \mathbf{1}$$

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$$\qquad\qquad\qquad \downarrow \qquad\qquad \downarrow \qquad\qquad\qquad\qquad \downarrow \qquad\qquad\qquad \downarrow$$

monopole: $\quad p_\alpha \qquad\quad x^\alpha \qquad\qquad\qquad p_\alpha \qquad\qquad\qquad x^i$

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Consider also the limit $R \to 0 \; (\rho \to 0)$:

$$[p_i, p_j, x^k] = \frac{\ell_{\mathrm{P}}^2}{2} \, \varepsilon_{ij}{}^k \, x^4 + \frac{\hbar \, \ell_{\mathrm{P}}}{2} \left(\delta_j^k \, x_i - \delta_i^k \, x_j \right) \, , \qquad [p_i, p_j, x^4] = -\frac{\ell_{\mathrm{P}}^3}{2\hbar} \, \varepsilon_{ijk} \, x^k \, ,$$

$$[p_i, p_j, p_k] = -2 \, \hbar \, \ell_{\mathrm{P}} \, \varepsilon_{ijk} \, p_4 \, , \qquad [p_4, p_i, x^j] = -\frac{\hbar \, \ell_{\mathrm{P}}}{2} \, \delta_i^j \, x^4 - \frac{\ell_{\mathrm{P}}^2}{2} \, \varepsilon_i{}^{jk} \, x_k \, ,$$

$$[p_4, p_i, x^4] = -\frac{\ell_{\mathrm{P}}^3}{2\hbar} \, x_i \, , \qquad [p_4, p_i, p_j] = -\frac{\ell_{\mathrm{P}}^2}{2} \, \varepsilon_{ijk} \, p^k \, ,$$

This algebra can be considered as the free, non-associative, eight-dimensional phase space algebra of M-theory.

It becomes trivial in the limit $l_P \to 0$.

Can one also get the full Spin(7) algebra with $R, \rho \neq 0$?

This seems to be possible in decompactification limit:

$$\frac{1}{V} R^{\mu, \nu\rho\alpha\beta} \, p_\mu = 0 \qquad \text{e.g.} \qquad \frac{R \, p_4}{V} = 0$$

$$V \to \infty \quad \Rightarrow \quad R \neq 0 \, , \quad p_4 \neq 0$$

Full Spin(7) algebra:

$$[x^i, x^j, x^k] = -\frac{\ell_s^3}{2} R \, \varepsilon^{ijk} \, x^4 \ , \qquad [x^i, x^j, x^4] = \frac{\lambda^2 \, \ell_s^3}{2} R \, \varepsilon^{ijk} \, x_k \ ,$$

$$[p^i, x^j, x^k] = -\frac{\lambda^2 \, \ell_s^3}{2} R \, \varepsilon^{ijk} \, p_4 - \frac{\lambda \, \ell_s^3}{2} R \left(\delta^{ij} \, p^k - \delta^{ik} \, p^j \right) \ ,$$

$$[p_i, x^j, x^4] = -\frac{\lambda^2 \, \ell_s^3}{2} R \, \delta_i^j \, p_4 - \frac{\lambda^2 \, \ell_s^3}{2} R \, \varepsilon^{ijk} \, p_k \ ,$$

$$[p_i, p_j, x^k] = \frac{\hbar^2 \, \lambda^2}{2} \, \varepsilon_{ij}{}^k \, x^4 + \frac{\hbar^2 \, \lambda}{2} \left(\delta_j^k \, x_i - \delta_i^k \, x_j \right) \ ,$$

$$[p_i, p_j, x^4] = -\frac{\hbar^2 \, \lambda^3}{2} \, \varepsilon_{ijk} \, x^k \ , \qquad [p_i, p_j, p_k] = -2 \, \hbar^2 \, \lambda \, \varepsilon_{ijk} \, p_4 \ ,$$

$$[p_4, x^i, x^j] = \frac{\lambda \, \ell_s^3}{2} R \, \varepsilon^{ijk} \, p_k \ , \qquad [p_4, x^i, x^4] = -\frac{\lambda^2 \, \ell_s^3}{2} R \, p^i \ ,$$

$$[p_4, p_i, x^j] = -\frac{\hbar^2 \, \lambda}{2} \, \delta_i^j \, x^4 - \frac{\hbar^2 \, \lambda^2}{2} \, \varepsilon_i{}^{jk} \, x_k \ ,$$

$$[p_4, p_i, x^4] = -\frac{\hbar^2 \, \lambda^3}{2} \, x_i \ , \qquad [p_4, p_i, p_j] = -\frac{\hbar^2 \, \lambda^2}{2} \, \varepsilon_{ijk} \, p^k \ ,$$

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V) Outlook & open questions

Non-associative algebras occur in M-theory at many places:

- Multiple M2-brane theories and 3-algebras

  J. Bagger, N. Lambert (2007)

- $Spin(7), G_2$ backgrounds

Generalization to higher dimensional exceptional field theory? Work in progress by D.L., E. Malek, M. Syväri

Quantization:

D. Mylonas, P. Schupp, R.Szabo, arXiv:1207.0926, arXiv:1312.1621

I. Bakas, D.L., arXiv:1309.3172

P. Aschieri, R.Szabo, arXiv:1504.03915

$$[x, x] \sim R \, p \quad \Rightarrow \quad \star_p - \text{product}$$

$$[x, x, x] \sim R \quad \Rightarrow \quad \triangle - \text{product}$$

$\Rightarrow$ Non-associative quantum mechanics

Volume quantization.

Interesting proposal for the quantization of non-geometric M-theory background by deriving a phase space star product for the non-associative algebra of octonions.

V. Kupriyanov, R.Szabo, arXiv:1701.02574:

Donnerstag, 6. Juli 17

Many thanks !