Logarithmic Corrections to Black Hole Entropy

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Corrections to Black Hole Entropy

- Agenda: *study quantum corrections* to black hole entropy *far from the BPS limit*.

- Methods:
  - *Explicit computation*.
  - Study *symmetries in effective quantum field theory*.

- Conventional wisdom: quantum corrections far from the BPS limit are large and complicated.

- Results:
  - Generally the corrections are complicated, as expected.
  - However, they greatly simplify in some *environments*.
Environmental Dependence

- Example: consider a **Kerr-Newman black hole** as a **solution to the Einstein-Maxwell theory**

\[
\mathcal{L} = \frac{1}{16\pi G_N} \left( R - \frac{1}{4} F_{\mu\nu} F^{\mu\nu} \right)
\]

- Now add a field that appears only quadratically in the action (such as a fermion \(\psi\)).

- We can study the “same” solution (unchanged **geometry and gauge field**): assume the additional field vanishes \(\psi = 0\).

- Environmental dependence: **quantum corrections depend on such additional fields** (for example, they run in loops).

- The upshot: **Kerr-Newman black holes simplify in an environment with \(\mathcal{N} \geq 2\) supersymmetry.**
This Talk

• *Embedding* of Kerr-Newman black holes into theories with \( \mathcal{N} \geq 2 \) SUSY.

• *Quantum corrections* to black hole entropy: explicit computation.

• A *non-renormalization theorem*.

Collaborators: Anthony M. Charles and Daniel R. Mayerson.

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N=2 SUGRA

- The baseline: $\mathcal{N} = 2$ minimal SUGRA has bosonic part identical to Einstein-Maxwell theory.

- Any bosonic solution remains a solution after the two gravitini are added because fermions can be consistently set to zero.

- More challenging: **couple to $n_V$ N=2 vector multiplets:**

\[
\mathcal{L} = \frac{1}{2\kappa^2} R - g_{\alpha\bar{\beta}} \nabla^\mu z^\alpha \nabla_\mu \bar{z}^{\bar{\beta}} + \frac{1}{2} \text{Im} \left[ \mathcal{N}_{IJ} F_{\mu\nu}^{+I} F^{+\mu\nu J} \right]
\]

- Comments:

  - Complex scalar fields in vector multiplets: $z^\alpha, \alpha = 1, \ldots, n_V$.
  
  - Vector fields $A^I_{\mu}$ include the graviphoton so $I = 0, \ldots, n_V$ (one more value).

  - Kähler metric $g_{\alpha\bar{\beta}}$ and **vector couplings** $\mathcal{N}_{IJ}$ depend on **scalars** as specified by special geometry.
Adding Scalars to Kerr-Newman

• Kerr-Newman does not have scalars so to maintain the “same” solution we take the $\mathcal{N} = 2$ scalars constant.

• An obstacle: generally the vector fields source the scalars so they cannot be constant.

• Solution: first specify the projective coordinates $X^I$ for the scalars, then specify the $\mathcal{N} = 2$ vectors ($F^{I}_{\mu\nu}$) in terms of the Einstein-Maxwell field strength ($F_{\mu\nu}$) and the scalars as:

$$F^{+I}_{\mu\nu} = X^I F^+_{\mu\nu}.$$ 

• Then the sources on the scalars cancel so it is consistent to have constant scalars.
General Embedding: the Upshot

- We consider *all theories with* $\mathcal{N} \geq 2$ *SUSY*.

- Summarize the matter content in terms of $\mathcal{N} = 2$ fields:
  - one SUGRA multiplet
  - $\mathcal{N} - 2$ (massive) gravitini
  - $n_V$ vector multiplets
  - $n_H$ hyper multiplets

- This *decomposition is useful for both BPS and non-BPS*.

- Our embedding takes the geometry unchanged, matter fields “minimal”, and guarantees that all equations of motion of $\mathcal{N} \geq 2$ SUGRA are satisfied.
Quantum Corrections: Generalities

- The entropy of a large black hole allows the expansion:

\[ S = \frac{A}{4G} + \frac{1}{2} D_0 \log A + \ldots . \]

- **Take all parameters with dimension length large**: so area \( A \sim (2MG)^2 \) (up to dimensionless ratios \( J/M^2 \) and \( Q/M \)).

- The logarithmic correction: \( \log A \sim \log 2MG \) up to the coefficient \( D_0 \), a nontrivial function of dimensionless ratios \( J/M^2 \) and \( Q/M \).

- The area \( A \) and the coefficient \( D_0 \) can both be **computed from the low energy theory**: only massless fields contribute.

- They each offer an **infrared window into the ultraviolet theory**.
Quantum Fluctuations: Strategy

- All contributions from \textit{quadratic fluctuations} around the classical geometry take the schematic form
  \[
e^{-W} = \int \mathcal{D}\phi \ e^{-\phi\Lambda\phi} = \frac{1}{\sqrt{\det\Lambda}}.
  \]
- The quantum corrections are encoded in the heat kernel
  \[
  D(s) = \text{Tr} \ e^{-s\Lambda} = \sum_i e^{-s\lambda_i}.
  \]
- The effective action becomes
  \[
  W = - \int_{\epsilon^2}^{\infty} \frac{ds}{2s} D(s) = - \int_{\epsilon^2}^{\infty} \frac{ds}{2s} \int d^Dx K(s) .
  \]
- The leading corrections are encoded in the \textit{s-independent term in} \( D(s) \) (denoted \( D_0 \)).
- Disclosure: zero modes are suppressed in this talk.
Aside: the Trace Anomaly

- The coefficient $D_0$ is closely related to the integrated trace anomaly.

- Gravity is far from conformal so there is no trace anomaly.

- Classically the effective action satisfies the scaling relation

  \[
  \left( M \frac{\partial}{\partial M} - 2 \right) \Gamma = 0
  \]

  so the classical on-shell action scales as $\Gamma \sim M^2 \sim A$

- We study the quantum effective action that satisfies the anomalous scaling relation

  \[
  \left( M \frac{\partial}{\partial M} - 2 + D_0 \right) \Gamma =
  \]
Interactions

- *In principle*: computations are straightforward applications of techniques from the 70’s.

- But: our embedding into SUGRA gives *nonminimal couplings*.

- For example, for fermions in $\mathcal{N} = 2$ hypermultiplets the background EM-field enters through *Pauli couplings*

  $$
  \mathcal{L}_{\text{hyper}} = -2\bar{\zeta} A \gamma^\mu D_\mu \zeta^A - \frac{1}{2} \left( \bar{\zeta} A F_{\mu\nu} \Gamma^{\mu\nu} \zeta^B \epsilon_{AB} + \text{h.c.} \right).
  $$

- Such nonminimal couplings force us to compute some new heat kernels.

- Schematic scaling in gravity: $[D + F, D + F] \sim R + F^2$. Like Einstein’s equation but unlike typical QFTs (minimal couplings).
Example

- Lagrangian for hyperfermion in the $\mathcal{N} = 2$ SUGRA multiplet:

$$\mathcal{L}_{\text{hyper}} = -2\bar{\zeta}_A \gamma^\mu D_\mu \zeta^A - \frac{1}{2} \left( \bar{\zeta}^A F_{\mu\nu} \Gamma^{\mu\nu} \zeta^B \epsilon_{AB} + \text{h.c.} \right).$$

Recall: there is Pauli coupling involving field strength $F_{\mu\nu}$.

- Heat kernel coefficient (computed using standard technology):

$$(4\pi)^2 a_4^{\text{hyper}}(x) = -\frac{1}{360} \left( -7R_{\mu\nu\rho\sigma} R^{\mu\nu\rho\sigma} - 232 R_{\mu\nu} R^{\mu\nu} - \frac{45}{4} (F^{\mu\nu} F_{\mu\nu})^2 ight.$$

$$+ \frac{45}{4} (F^{\mu\nu} \tilde{F}_{\mu\nu})^2 \right).$$

- Consistent with scaling $R \sim F^2$,

- Generally there are terms of schematic form $R^2, RF^2, F^4$ (allowed by scaling symmetry).
General form of 2nd Seeley-deWitt coefficient:

\[ a_4(x) = \alpha_1 R_{\mu\nu\rho\sigma} R^{\mu\nu\rho\sigma} + \alpha_2 R_{\mu\nu} R^{\mu\nu} + \alpha_3 R_{\mu\nu\rho\sigma} F^{\mu\nu} F^{\rho\sigma} + \ldots \]

After simplifications using Einstein equation, Bianchi identities, ....

\[ a_4(x) = \frac{c}{16\pi^2} W_{\mu\nu\rho\sigma} W^{\mu\nu\rho\sigma} - \frac{a}{16\pi^2} E_4, \]

Euler density

\[ E_4 = R_{\mu\nu\rho\sigma} R^{\mu\nu\rho\sigma} - 4R_{\mu\nu} R^{\mu\nu} + R^2. \]

Note: all dependence on field strength can be traded for curvature terms.

Final results can be expressed in terms of \( c, a \) only!
Duality: Einstein-Maxwell Theory

• **Why** can all explicit dependence on field strength be eliminated?

• Electromagnetic duality requires that *four derivative terms are duality invariant* (even though two derivative terms are not).

• A unique duality invariant tensor: \( \mathcal{I}_{\mu\nu\rho\sigma} = F^{+\mu\nu} F^{-\rho\sigma} \)

• All Lorentz invariants (eg. \( \mathcal{I}_{\mu\nu\rho\sigma} \mathcal{I}^{\mu\nu\rho\sigma} \)) can be recast in terms of:

  \[
  \mathcal{I}^{\rho}_{(\mu\nu)\rho} = F^{+\rho}_{\mu\nu} F^{-\rho}_{\rho\nu} = R_{\mu\nu}
  \]

• Upshot: duality precludes explicit dependence on \( F_{\mu\nu} \) so

  *anomaly coefficients expressible in terms of geometry alone.*
Duality: $\mathcal{N} = 2$ Supergravity

- Duality in $\mathcal{N} = 2$ supergravity: symplectic invariance

- Embedding shows that the Maxwell field $F^+_{\mu\nu}$ is duality invariant

$$F^{+I}_{\mu\nu} = X^I F^+_{\mu\nu}. \quad (1)$$

- $U(1)_R$ symmetry: $F^{+I}_{\mu\nu}$ neutral, $X^I$ charged, so $F^+_{\mu\nu}$ is (negatively) charged.

- Electromagnetic duality symmetry of Einstein-Maxwell descends from $U(1)_R$ symmetry of $\mathcal{N} = 2$ supergravity.

- Upshot: $U(1)_R$ symmetry precludes explicit dependence on $F_{\mu\nu}$, anomaly coefficients expressible in terms of geometry alone.
Results: Logarithms from Bosons

- Contributions from bosons in $\mathcal{N} \geq 2$ theory:

  \[
  c^{\text{boson}} = \frac{1}{60} \left( 137 + 12(\mathcal{N} - 2) - 3n_V + 2n_H \right)
  \]

  \[
  a^{\text{boson}} = \frac{1}{90} \left( 106 + 31(\mathcal{N} - 2) + n_V + n_H \right)
  \]

- The bosons in the $n_H$ hyper multiplets and $\mathcal{N} = 2$ gravitino multiplets are minimally coupled so these values for $a, c$ are standard.

- The bosons in the $n_V$ vector multiplets and the supergravity multiplet couple to the field strength so, after eliminating $F^2$ in favor of $R$, these values of $a, c$ are nonstandard.

- For fermions the situation is reversed.
Integrals

- General form of quantum corrections to the entropy:

\[ \delta S = \frac{1}{2} D_0 \left( \frac{Q}{M}, \frac{J}{M^2} \right) \log A_H \]

- The \textit{a-term gives a universal contribution} to \( D_0 \) (independent of black hole parameters) because

\[ \chi = \frac{1}{32\pi^2} \int d^4x \sqrt{-g} E_4 = 2 \] .

- The \textit{c-term gives a complicated contribution} to \( D_0 \):

\[ \int d^4x \sqrt{-g} W_{\mu\nu\rho\sigma} W^{\mu\nu\rho\sigma} = 64\pi^2 + \frac{\pi \beta Q^4}{b^5 r_H^4 (b^2 + r_H^2)} \left[ 4b^5 r_H + 2b^3 r_H^3 ight. \\
+ \left. 3(b^2 - r_H^2)(b^2 + r_H^2)^2 \tan^{-1} \left( \frac{b}{r_H} \right) + 3br_H^5 \right] . \]

\[ b = J/M, \quad r_H = M + \sqrt{M^2 - b^2}, \quad \beta = 1/T. \]
Logarithms from Fermions

• Contributions from bosons in $\mathcal{N} \geq 2$ theory:
  
  $$c_{\text{boson}} = \frac{1}{60} (137 + 12(\mathcal{N} - 2) - 3n_V + 2n_H)$$
  
  $$a_{\text{boson}} = \frac{1}{90} (106 + 31(\mathcal{N} - 2) + n_V + n_H)$$

• Contributions from fermions in $\mathcal{N} \geq 2$ theory:
  
  $$c_{\text{fermion}} = \frac{1}{60} (-137 - 12(\mathcal{N} - 2) + 3n_V - 2n_H)$$
  
  $$a_{\text{fermion}} = \frac{1}{360} (-589 + 41(\mathcal{N} - 2) + 11n_V - 19n_H)$$

• The $c$ coefficient vanishes in $\mathcal{N} \geq 2$ theory!

• A huge simplification: Weyl$^2$ terms are complicated in general backgrounds.

• It is a surprise: SUSY of the background $\Rightarrow$ AdS$_2 \times S^2 \Rightarrow$ Weyl$^2 = 0 \Rightarrow$ vanishing coefficient of Weyl$^2$ not noticed.
Higher Derivative Corrections

- *Why* is the anomaly coefficient $c = 0$?

- Background is generally not supersymmetric so *fluctuations are not organized in supermultiplets*.

- Background field formalism realizes symmetry explicitly: dependence on background fields respect $\mathcal{N} = 2$ supersymmetry.

- Schematic form of effective action

$$\mathcal{L}_4 = g_W(Weyl^2 + \text{SUSY partners}) - g_E(\text{Euler} + \text{SUSY partners})$$

Coefficients $g_W, g_E$ are *running couplings* with $\beta$-function related to $c, a$. 
Higher Derivatives and $\mathcal{N} = 2$ SUSY

- Details of the action: off-shell formalism from reduction of superconformal supersymmetry, a lot of auxiliary fields,...... (details involve hard work).

- SUSY completion of $\text{Weyl}^2$ has been known for a long time.

- Schematic of on-shell structure:

$$\text{Weyl}^2 + \text{SUSY partners} = E_4$$

Cartoon: there is an elaborate cancellation between gravitational terms ($\text{Weyl}^2$), their matter partners ($F^4$), and cross-terms ($RF^2$).
Higher Derivatives and $\mathcal{N} = 2$ SUSY

• SUSY completion of $E_4$ was identified only in the last few years.

• Schematic of on-shell structure:

\[ E_4 + \text{SUSY partners} = E_4 \]

Cartoon: the matter terms vanish on-shell.

• So: all matter terms can be eliminated in favor of geometry alone.

• And: both four-derivative $\mathcal{N} = 2$ invariants reduce to the Euler invariant $E_4$.

• The anomaly $c = 0$ because $W^2$ is inconsistent with $\mathcal{N} = 2$ SUSY.
Summary

• Logarithmic corrections to black hole entropy in $\mathcal{N} \geq 2$ SUGRA simplify greatly even when BHs preserve no SUSY.

• The coefficient is universal: it depends only on the theory (not on parameters of the black hole)

$$\delta S = \frac{1}{12} \left( 23 - 11(\mathcal{N} - 2) - n_V + n_H \right) \log A_H .$$

• These corrections can be reproduced from microscopic theory in some BPS cases.

• The IR theory is a window into the UV theory: apparently the deformation (far!) off extremality is independent of coupling!
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