Effective Action from M-theory on twisted connected sums

Ascona Monte Verita, 6 July 2017

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arXiv:1702.05435, with Thaisa Guio, Hans Jockers and Hung-Yu He
Motivation

1. 11D supergravity: An unique set up
2. $G_2$ manifolds: An unique compactification class
3. Yet many questions

Constructing $G_2$ Manifolds

1. Structure theorems for $SU_n$ and $G_2$ holonomy mflds
2. The twisted gluing construction
3. Examples

The effective action

1. Homology and Spectrum
Kähler potential, gauge kinetic function, superpotential

The Kovalon

$\mathcal{N} = 2$ and $\mathcal{N} = 4$ sectors

Transitions between $G_2$ manifolds

Abelian Gauge symmetry, charged matter spectrum

Non-abelian Gauge symmetry, charged matter spectrum

Conclusions
Motivation:

1. M-theory: An unique set up

D.o.f. spinors $2\left\lfloor \frac{D+1}{2} \right\rfloor - 1$, d.o.f. of bosons $p_r(D)$. So there is a maximal dimension $D$ for super symmetric representations.

Supersymmetry representation of physical theories singles out especially two cases W. Nahm 1978.

- Eleven dimensional supergravity $D = 11$ ✓
- Six dimensional superconformal field theory
The action of eleven dimensional supergravity was found by Cremmer, Julia, Scherk 1978

\[ S_{11d} = \frac{1}{2\kappa_{11}^2} \int \left( \ast R_S - \frac{1}{2} d\hat{C} \wedge \ast d\hat{C} - \ast i \bar{\Psi}_M \hat{\Gamma}^{MNP} \hat{D}_N \hat{\Psi}_P \right) \]

\[ - \frac{1}{192\kappa_{11}^2} \int \ast \bar{\Psi}_M \hat{\Gamma}^{MNPQRS} \Psi_N (d\hat{C})_{[PQRS]} - \frac{1}{2\kappa_{11}^2} \int d\hat{C} \wedge \ast_{11} \hat{F} \]

\[ - \frac{1}{12\kappa_{11}^2} \int d\hat{C} \wedge d\hat{C} \wedge C + \ldots , \]

Here \( \kappa_{11} = \hat{G}_N, \hat{\Psi} \) the gravitino, \( \hat{C} \) anti-symmetric 3-form, \( \hat{F}_{[MNPQ]} = 3\hat{\Psi}_{[M} \hat{\Gamma}^{NP} \hat{\Psi}_{Q]}, + \ldots 4\)-fermion int.
Simple unique beautiful starting point for K-K reduction

2 G2 manifolds: To get $\mathcal{N} = 1$ sugra in 4d, look in Berger’s list of special holonomy manifolds. Beyond the generic cases, ∃ two entries

- (vi) $d = 7$: $\text{Hol}(g) = G_2$, $G_2$-mfld, Ricci-flat $R_{ij}(g) = 0$, $\mathcal{N} = 1$ covariant constant spinor; $\varphi$ associative 3-form, $\ast\varphi$ coassociative 4-form.

- (vii) $d = 8$: $\text{Spin}(7)$ : $N_+ = 1$, $\psi$ Cayley 4-form

That is we get an unique K-K compactification, supposingly unifying non-perturbative String theories
Yet many questions:

• 1.) How to construct compact $G_2$ manifolds?

• 2.) How to geometrical engineer the ones that yield interesting $\mathcal{N} = 1$ supergravities including the standard model?

• 3.) How to calculate Kaluza-Klein and M-theory corrections to the effective $\mathcal{N} = 1$ supergravity action?

• 4.) How those relate to other $\mathcal{N} = 1$ vacua?
Constructing $G_2$ Manifolds

1 Structure theor. for $SU_n$ and $G_2$ holonomy mflds: To address 1.) recall the more familiar $N = 2$ situation of CY manifolds $X$ ($n = 3$)

- (iii) $d = 2n$, $n \geq 2$: $\text{Hol}(g) = SU_n$, CY-mfld, Ricci-flat, Kähler, $N_\pm = 1$ for $n$ odd, $N_+ = 2$ for $n$ even, $\omega$ Kähler $(1,1)$-form and $\Omega$ hol.harm. $(n,0)$-form.

Existence Theorem of Yau

\[ K_X = -c_1(T_X) = 0 \rightarrow \exists \text{ unique}^1 g \text{ with } R_{ij}(g) = 0 \]

\[^1\text{Given complex– (}\Omega\text{) and Kähler structure (}\omega\text{)}\]
Controlling $K_X$ is trivial by multiplicative properties of Chern characters. Take a Fano variety such as $\mathbb{P}^n$ and $X_{n-1}$ as zero locus of a degree $d$ hypersurface. Then

\[
\text{ch}(T_X) = \frac{\text{ch}(T_{\mathbb{P}^n})}{\text{ch}(\mathcal{N}_X)} = \frac{(1 + H)^{n+1}}{1 + dH} = 1 + c_1(T_X) + \ldots
\]

\[
= 1 + [(n + 1) - d]H + \ldots, \quad \text{i.e.}
\]

$c_1(T_X) = 0 \iff d = (n + 1)$. **Strategy:**

$c_1(T_X) = 0 \rightarrow^1 R_{i\overline{j}}(g) = 0$, $(R_{i\overline{j}}(g) = 0 \& \pi_1(X) \text{ fin.}) \rightarrow \text{Hol}(g) = SU(n) \rightarrow N = 2 4d - \text{susy}$
Remarks:

- Slight generalisations $\mathbb{P}^n \rightarrow \mathbb{P}_{\Delta_n}$ with $(\Delta_n, \Delta^*_n)$ a pair of reflexive polyhedra, yields $10^8$ families of compact CY 3-folds and 4319 non-compact CY 3 folds.

- Similar as above one can define a non-compact CY as $X_n = \mathbb{P}^n \setminus \{P_{d=n+1}(x) = 0\}$. For an easy example think of $n = 1$, then $X_1 = \{S^2 \setminus 2 \text{ points}\} \sim \text{cylinder}$, which clearly allows a flat metric. In higher dimensions Tian & Yau established the existence of a no-where vanishing $\Omega$ — trivializing $K_X$ — together with a boundary asymptotic, so that Yau’s theorem still applies.
$G_2$ structure manifolds: $G_2$ is a 14d simply connected subgroup of $SO(7)$. Geometrically it arises as follows. \( \exists \varphi \in \Lambda_+^3(\mathbb{R}^7)^* \) a 3-form on $\mathbb{R}^7$ such that

\[
B_\varphi(X, Y) = -\frac{1}{3!} (X \hook\varphi) \wedge (Y \hook\varphi) \wedge \varphi \tag{1}
\]

is a positive define bilinear form with respect to an oriented volume form. $GL(7, \mathbb{R})$ acts on $\varphi$ and $G_2$ is its fourteen dimensional stabilizer group R.L. Bryant 1987.

A $G_2$ structure on an oriented 7d manifold $Y$ is a 3-form $\varphi$ which is $\forall p \in Y$ oriented isomorphic to $\Lambda^3 T_p^* Y \simeq \Lambda_+^3(\mathbb{R}^7)^*$. Via (1) this defines a Riemannian
metric on \( Y \)

\[
g_\varphi(X_p, Y_p) = \frac{B_\varphi(X_p, Y_p) (\partial_1|_p, \ldots, \partial_7|_p)}{\text{vol}_\varphi(\partial_1|_p, \ldots, \partial_7|_p)} ,
\]

Theorem Fernández & Gray 1983 \( Y \) has a holomomy \( \text{Hol}(g_\varphi) \subset G_2 \) iff

\[
d\varphi = 0, \quad d*_{g_\varphi} \varphi = 0 .
\]

\( \text{Hol}(g_\varphi) = G_2 \) iff \( \pi_1(Y) \) is finite.

- Note the non-linearity in harmonicity condition for \( \varphi \).
• A harmonic $G_2$ structure $\varphi$ is called torsion free in the sense that the Levi-Cevita connection has $G_2$ holonomy.

• “Strategy”: 
  1. Show existence of torsion free $\varphi$ on $Y$.
  2. Show that $\pi_1(Y)$ is finite.

• The first part has boring solutions: $Y_0 = X \times S^1$, with $\theta$ the angl. coord. on $S^1$ one has torsion free $G_2$ structure

\[
\varphi_0 = \gamma \, d\theta \wedge \omega + \text{Re}(\Omega), \quad *\varphi_0 = \frac{1}{2} \omega^2 - \gamma \, d\theta \wedge \text{Im}(\Omega)
\]

But $\pi_1(Y) = \mathbb{Z}$ and therefore $\rightarrow \text{SU}_3$ holonomy and
$N = 2$ 4d supergravity. ($\gamma \in \mathbb{R}$)

2 The twisted gluing construction


Alternative approach: First compact examples constructed by Dominic Joyce (1994) as $Y = \overset{\sim}{T^7}/G$ resolutions. ... Simons Collaboration
Basic idea of the twisted connected gluing:

- Construct two non-compact Calabi-Yau 3-fold as discussed by Tian and Yau, called $X_{L/R}$, where $L/R$ stands for Left and Right.

- Construct two product 7-folds $Y_{L/R} = X_{L/R} \times S^1_{L/R}$, with the “trivial” torsion free $G_2$-structures $\varphi_{0\ L/R}$.

- Each $X_{L/R}$ has a K3 called $S_{L/R}$ removed.

- There is another canonical $S^*_{L/R}$ in $\mathcal{N}_{S_{L/R}} \subset X_{L/R}$ parametrizing in polar coordinates $|z| = e^t$ and $\theta^*$ a disk.
\[ D_{L/R} \in X_{L/R}. \]

- Glue \( Y_L \) to \( Y_R \) to obtain \( Y \) so that
  - a) \( \varphi_0_{L/R} \) extend to a torsion free \( G_2 \) structure on \( Y \). This requires a hyperkähler rotation on the \( K3 \) boundaries
  - b) the infinite \( \pi_1(Y_{L/R}) \) becomes finite (\( \pi_1(Y) = 0 \)). This is achieved as in the Hopf gluing of two solid tori here \( D_{L/R} \times S^1_{L/R} \) to an \( S^3 \) with \( \pi_1(S^3) = 0 \).

By the structure theorem \( Y \) is then a manifold whose metric has the full \( G_2 \) holonomy.
Let us visualize this as good as we can:

Figure 1: Kovalev’s twisted connected sum construction.
Gluing the asymptotic regions: Before the $K3$ is cut out, one needs to remove its self-intersection $[S^2] = [C]$. That is achieved by blowing up along $C$. The asymptotic region near the $K3$, that is cut out, has hence a simple form, called asymptotically cylindrical Calabi-Yau 3-fold $X^\infty = S \times \Delta_{cyl}$, with $\Delta_{cyl} = \{ z \in \mathbb{C} | |z| > 1 \}$. In this region the Kähler- and the holomorphic 3-form are given by

\[
\omega^\infty = \gamma^* \frac{i dz \wedge d\bar{z}}{2 z \bar{z}} + \omega_S = \gamma^* \frac{2 dt \wedge d\theta^*}{\bar{z}} + \omega_S ,
\]
\[
\Omega^\infty = -\gamma^* \frac{idz}{z} \wedge \Omega_S = \gamma^* (d\theta^* - idt) \wedge \Omega_S .
\]
Here \((\omega_S, \Omega_S)\) are Kähler- and holomorphic two-form of \(S\), \(z = e^{t+i\theta^*}\) and \(\gamma^*\) the length scale of \(\Delta^\text{cyl}\). Let \(K\) be the compact part of \(X\) then there is a diffeomorphism \(\eta : X^\infty \rightarrow X \setminus K\) so that

\[
\eta^* \omega - \omega^\infty = d\mu \quad \text{with} \quad |\nabla^k \mu| = O(e^{-\lambda \gamma^* \xi}) ,
\]

\[
\eta^* \Omega - \Omega^\infty = d\nu \quad \text{with} \quad |\nabla^k \nu| = O(e^{-\lambda \gamma^* \xi}) ,
\]

with \(\lambda = \min \left\{ \frac{1}{\gamma^*}, \lambda_S \right\}\). Here \(\lambda_S\) is the smallest positive eigenvalue of \(\nabla^2_S\). This ensures that one can glue the asymptotic forms (2), which are fast enough approximated to yield a torsion free \(\varphi\) on \(Y\) as follows:
On the gluing region
\[ Y_{L/R}^\infty = X_{L/R}^\infty \times S_{L/R}^1 = S_{L/R} \times \Delta_{L/R}^{cyl} \times S_{L/R}^1, \]
we define
\[ \omega_{S_{L/R}}^\infty = \omega_{L/R}^I, \quad \Omega_{S_{L/R}}^\infty = \omega_{L/R}^J + i \omega_{L/R}^K. \]

We get then on \( Y_{L/R}^\infty \) a torsion free 3-form
\[
\varphi_{0 L/R}^\infty = \gamma_{L/R} d\theta_{L/R} \wedge (\gamma_{L/R}^* 2 dt_{L/R} \wedge d\theta^*_{L/R} + \omega_{S_{L/R}}^\infty) \\
+ \gamma_{L/R}^* d\theta^*_{L/R} \wedge \text{Re}(\Omega_{S_{L/R}}^\infty) + \gamma_{L/R}^* dt_{L/R} \wedge \text{Im}(\Omega_{S_{L/R}}^\infty). 
\]

The gluing diffeomorphism: First we need the \( K3 \) to be isometric with respect to a hyperkähler rotation.
Then there is a family \((\Lambda \in \mathbb{R})\) of gluing diffeomorphisms defined as Kovalev (2003)

\[ F_\Lambda : (\theta^*_L, t_L, u^\alpha_L, \theta_L) \mapsto (\theta^*_R, t_R, u^\alpha_R, \theta_R) = (\theta_L, \Lambda - t_R, r(u^\alpha_L), \theta_L) \]

\((\theta^*_{L/R}, t_{R/L})\) of \(\Delta^\text{cyl}_{L/R}\), \(u^\alpha_{L/R}\) coords of \(S_{L/R}\), and \(\theta_{L/R}\) of \(S^1_{L/R}\). With \(\gamma := \gamma_L = \gamma_R = \gamma^*_L = \gamma^*_R\) it is easy to check that

\[ F^*_\Lambda \varphi_{0_R} = \varphi_{0_L} . \]
With $X_{L/R}(T) = K_{L/R} \cup \eta_{L/R}(\mathbb{R}^{<T+1})$, $Y_{L/R}(T) = X_{L/R}(T) \times S^1_{L/R}$

$Y = Y_L(T) \cup_{F_{2T+1}} Y_R(T)$.

Kovalev’s checks that $Y$ is a $G_2$ manifold in two steps: First he establishes analytically that

$$\tilde{\omega}_{L/R}^T = \omega_{L/R} - d(\alpha(t - T)\mu_{L/R})$$,
$$\tilde{\Omega}_{L/R}^T = \Omega_{L/R} - d(\alpha(t - T)\nu_{L/R})$$,
define an interpolating $G_2$ structure on $Y_{L/R}$ as

$$\tilde{\varphi}_{L/R}(\gamma, T) = \gamma \, d\theta \wedge \tilde{\omega}_{L/R}^T + \text{Re}(\tilde{\Omega}_{L/R}^T),$$

which extend to a torsion free $G_2$ structure on $Y$, because the latter can be approximated as

$$\varphi(\gamma, T) = \tilde{\varphi}(\gamma, T) + d\rho(\gamma, T) \quad \text{with} \quad |\nabla^k \rho(\gamma, T)| = O(e^{-\gamma \lambda T}),$$

in terms of the norm $|\cdot|$ and the Levi–Civita connection $\nabla$ of the metric induced from the asymptotic $G_2$-structure.
Secondly he shows topologically that

\[ \pi_1(Y) = \pi_1(X_L) \times \pi_1(X_R) \]

which completes his proof that \( Y \) has full \( G_2 \) holonomy \( \bullet \)
Fibration Structures: $Y_{L/R}$ is $K3$ fibrations over solid tori

$T_{L/R} \equiv S^1_{L/R} \times D_{L/R}$

The gluing of two solid tori $T_{L/R}$ to an $S^3$ induces the fibration.
Examples As we already mentioned there are 4319 examples of non-compact CY-folds $X$ constructed in weak Fano toric ambient spaces. To easily establish the isometric map $r : S_L \rightarrow S_R$ one needs the technical condition of semi ample canonical class (semi Fano), which leaves 899 $\mathbb{P}_\Delta$, for which $\Delta$ contains no codim 2-points lattice points. The latter have different Kähler cones, so roughly one gets $10^8 \times m_g$ examples, where $m_g$ is the gluing multiplicity of order $O(10)$.

They have to be distinguished by their Betti numbers. Using the Mayer-Vietoris sequence of the gluing map one
gets these topological data as

\[ H^2(Y, \mathbb{Z}) \simeq (k_L \oplus k_R) \oplus (N_L \cap N_R) \]
\[ H^3(Y, \mathbb{Z}) \simeq H^3(Z_L, \mathbb{Z}) \oplus H^3(Z_R, \mathbb{Z}) \oplus k_L \oplus k_R \oplus N_L \cap T_R \]
\[ \oplus N_R \cap T_L \oplus \mathbb{Z}[S] \oplus L/ (N_L + N_R) \]

Here \([S]\) is the Poincaré dual three-form of a the K3 fibre \(S\) in \((Z_{L/R}, S_{L/R})\), \(L \simeq H^2(S_L, \mathbb{Z}) \simeq H^2(S_R, \mathbb{Z})\). The inclusion \(\rho_{L/R} : S_{L/R} \hookrightarrow X_{L/R}\) induce maps \(\rho^*_{L/R} : H^2(X_{L/R}, \mathbb{Z}) \to L\) defining kernels \(k_{L/R} := \ker \rho^*_{L/R}\), images \(N_{L/R} := \text{Im} \rho^*_{L/R}\), and the transc. lattices \(T_{L/R} = N_{L/R}^\perp = \{l \in L \mid \langle l, N_{L/R} \rangle = 0\}\).
Orthonogal gluing examples:

\[ W = N_L + N_R = N_L \perp_R N_R, \quad R = N_L \cap N_R \quad \text{and} \quad N_L \perp R \subset N_R, \quad N_R \perp L \subset N_L. \]

Building blocks

<table>
<thead>
<tr>
<th>No.</th>
<th>( \text{rk} N )</th>
<th>(-K^3)</th>
<th>( \kappa )</th>
<th>( e )</th>
<th>( e^2 )</th>
<th>( b_3(Z) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>MM27, K62 (Fano)</td>
<td>3</td>
<td>48</td>
<td>( \begin{pmatrix} 0 &amp; 2 &amp; 2 \ 2 &amp; 0 &amp; 2 \ 2 &amp; 2 &amp; 0 \end{pmatrix} )</td>
<td>( \begin{pmatrix} 1 \ 0 \ -1 \end{pmatrix} )</td>
<td>-4</td>
<td>50</td>
</tr>
<tr>
<td>MM25, K68 (Fano)</td>
<td>3</td>
<td>44</td>
<td>( \begin{pmatrix} 0 &amp; 2 &amp; 1 \ 2 &amp; 0 &amp; 3 \ 1 &amp; 3 &amp; -2 \end{pmatrix} )</td>
<td>( \begin{pmatrix} -1 \ 1 \ 0 \end{pmatrix} )</td>
<td>-4</td>
<td>46</td>
</tr>
<tr>
<td>MM31, K105 (Fano)</td>
<td>3</td>
<td>52</td>
<td>( \begin{pmatrix} 0 &amp; 2 &amp; 1 \ 2 &amp; 0 &amp; 3 \ 1 &amp; 3 &amp; -2 \end{pmatrix} )</td>
<td>( \begin{pmatrix} -1 \ 1 \ 0 \end{pmatrix} )</td>
<td>-4</td>
<td>54</td>
</tr>
</tbody>
</table>

Table 1: Data of low rank toric terminal Fano threefolds. \( R \) generated by a vector of length square \(-4\).
The formulas for the Betti numbers simplify for the orthogonal gluing

\[ b_2(Y) = \text{rk } R + \dim k_L + \dim k_R, \]
\[ b_3(Y) = b_3(Z_L) + b_3(Z_R) + \dim k_L + \dim k_R - \text{rk } R + 23 \]
The effective action

Homology and Spectrum:

Massless four-dimensional modes arise from the coefficients in the decomposition of the eleven-dimensional anti-symmetric three-form tensor $\hat{C}$ as

$$\hat{C}(x, y) = \sum_I A^I(x) \wedge \omega^{(2)}_I(y) + \sum_i P^i(x) \wedge \rho^{(3)}_i(y),$$

in terms of the harmonic two-forms $\omega^{(2)}_I$ and three-forms $\rho^{(3)}_i$ identified with non-trivial cohomology representatives of $H^2(Y)$ and $H^3(Y)$ of dimension $b_2(Y)$ and $b_3(Y)$,
respectively. In the paper further details about the
dimensional reduction of the fermions can be found. The
spectrum is summarized as follows:

<table>
<thead>
<tr>
<th>Multiplicity</th>
<th>Massless 4d component fields</th>
<th>Massless 4d</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>bosonic fields</td>
<td>fermionic fields</td>
</tr>
<tr>
<td>1</td>
<td>metric ( g_{\mu\nu} )</td>
<td>gravitino ( \psi_\mu, \psi^*_\mu )</td>
</tr>
<tr>
<td>( i = 1, \ldots, b_3(Y) )</td>
<td>scalars ( (S^i, P^i) )</td>
<td>spinors ( \chi^i, \chi^{*i} )</td>
</tr>
<tr>
<td>( I = 1, \ldots, b_2(Y) )</td>
<td>vectors ( A^I_\mu )</td>
<td>gauginos ( \lambda^I_\alpha )</td>
</tr>
</tbody>
</table>

Note that on a smooth \( G_2 \) manifold there are neither
non-abelian Gauge groups nor charged matter.
Kähler potential, gauge kinetic function, superpotential: The dimensional reduction of the Einstein–Hilbert term and the three-form tensor $\hat{C}$ yields the four-dimensional action Beasley Witten (2002db)

$$S_{4d}^{\text{bos}} = \frac{1}{2\kappa_4^2} \int \left[ \star_4 R_S + \frac{\kappa_{IJk}}{2V_{Y_0}} \left( S^k F^I \wedge \star_4 F^J - P^k F^I \wedge F^J \right) \right. \\
\left. - \frac{7}{2V_{Y_0}} \int_Y \rho_i^{(3)} \wedge g_{\varphi} \rho_j^{(3)} \left( dP^i \wedge \star_4 dP^j - dS^i \wedge \star_4 dS^j \right) \right]$$

in terms of the four-dimensional Hodge star $\star_4$, the Ricci scalar $R_S$, with respect to the metric $g_{\mu\nu}$, the reference volume $V_{Y_0}$, and the seven-dimensional Hodge star $\star_7$. 
From this we can derive the Kähler potential and the gauge kinetic coupling matrix

\[ K(\phi, \bar{\phi}) = -3 \log \left( \frac{1}{7} \int_Y \varphi \wedge \ast g_\varphi \varphi \right), \]
\[ f_{IJ}(\phi) = 2V_{Y_0} \sum_k \phi^k \int_Y \omega_I^{(2)} \wedge \omega_J^{(2)} \wedge \rho_k^{(3)} \]
\[ = 2V_{Y_0} \sum_k \kappa_{IJk} \phi^k, \]

as well as the superpotential

\[ W(\phi^i) = \frac{1}{4V_{Y_0}} \int_Y G \wedge (C + i\varphi). \]

\textbf{4.} \( \mathcal{N} = 2 \) and \( \mathcal{N} = 4 \) sectors: Note first that the Kovalev construction implies that there is a scale \( 2T + 1 \), which
separates $Z_L$ from $Z_L$. Taking this scale to $\infty$ one roughly expects the spectrum to separate into sectors.

<table>
<thead>
<tr>
<th>local geometry</th>
<th>multiplicity of $\mathcal{N} = 1$ multiplets</th>
<th>$U(1)$ vector multiplets</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$U(1)$ vectors</td>
<td>chirals</td>
</tr>
<tr>
<td>$Y_L = S^1_L \times X_L$</td>
<td>$\dim k_L$</td>
<td>$\dim k_L$</td>
</tr>
<tr>
<td>$SU(3)$ holonomy</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$Y_R = S^1_R \times X_R$</td>
<td>$\dim k_R$</td>
<td>$\dim k_R$</td>
</tr>
<tr>
<td>$SU(3)$ holonomy</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$T^2 \times S \times (0,1)$</td>
<td>$\dim N_L \cap N_R$</td>
<td>$3 \cdot \dim N_L \cap N_R$</td>
</tr>
<tr>
<td>$SU(2)$ holonomy</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

The Kovalon: More precisely one has to define of course complex $\mathcal{N} = 1$ neutral chiral moduli. We identify
two universal moduli: $\nu$ related to the overall volume and $\kappa$ related to the gluing parameters so that

$$\text{Re}(\nu) = v, \quad \text{Re}(\kappa) = vb.$$

Here $b$ is the squashing parameter of the $S^3$. We refer to the chiral multiplet $\kappa$ as the Kovalon, as it describes in the limit $\text{Re}(\kappa) \to +\infty$ — while keeping $\text{Re}(\nu)$ constant — the Kovalev limit.

The remaining real moduli fields are not universal and relate to the non-universal neutral chiral multiplets as

$$\text{Re}(\phi^\hat{i}) = v\tilde{S}^\hat{i}, \quad \text{Re}(\phi^\tilde{i}) = vb\tilde{S}^\tilde{i}.$$
They depend on the topological details of the building blocks \((Z_{L/R}, S_{L/R})\) and the choice of gluing diffeomorphism.

Analysing the gluing maps allows e.g. to determine the leading order dependence on the universal moduli

\[
K = -\log \left[ \left( V_{\tilde{S}}(\tilde{S}) \right)^3 (\nu + \bar{\nu})^4 (\kappa + \bar{\kappa})^3 + A(\tilde{S}, \nu + \bar{\nu}, \kappa + \bar{\kappa}) e^{-\lambda \frac{\kappa + \bar{\kappa}}{(\nu + \bar{\nu})^{1/3}}} \right],
\]

where the coefficient of the exponentially suppressed correction is expected to generically depends on both universal and non-universal geometric moduli fields.
Transitions between $G_2$ manifolds:

Generally one expects in $M$-theory on $G_2$ manifolds
gauge symmetry from codimension four singularities,
charged matter from codimension six singularities and
chiral spectra at codimension seven. We can realize the
first two phenomena on the building blocks using
essentially $\mathcal{N} = 2$ techniques. Our implications are that
there are Higgs transitions these sectors which are
compatible with the gluing diffeomorphism and lead to
genuine $G_2$ transitions.

Abelian Gauge symmetry, charged matter spectrum:
For the semi-Fano threefold $P$ we pick two global sections $s_0$ and $s_1$ of the anti-canonical divisor $-K_P$. However, instead of choosing a generic section $s_0$, we assume that the global section $s_0$ factors into a product

$$s_0 = s_{0,1} \cdots s_{0,n},$$

such that $s_{0,i}$ are global sections of line bundles $\mathcal{L}_i$ with $-K_P = \mathcal{L}_1 \otimes \cdots \otimes \mathcal{L}_n$. As a consequence the curve $\mathcal{C}_{\text{sing}} = \{s_0 = 0\} \cap \{s_1 = 0\}$ becomes reducible and
decomposes into

\[ C_{\text{sing}} = \sum_{i=1}^{n} C_i , \quad C_i = \{ s_{0,i} = 0 \} \cap \{ s_1 = 0 \} , \]

where we assume that the individual curves \( C_i \) are smooth and reduced. Following Kovalev and Lee 2008, we construct the building block \((Z^\#, S)\) associated to \( P \) by the sequence of blow-ups \( \pi_{\{C_1, \ldots, C_n\}} : Z^\# \to P \) along the individual curves \( C_i \) according to

\[ Z^\# = \text{Bl}_{\{C_1, \ldots, C_n\}} P = \text{Bl}_{C_n} \text{Bl}_{C_{n-1}} \cdots \text{Bl}_{C_1} P . \]

Since the curves \( C_i \) and the semi-Fano threefold \( P \) are
smooth, the blow-up $Z^\#$ is smooth as well. As before, the
K3 surface $S$ arises as the proper transform of a smooth
anti-canonical divisor $S^\# = \{ \alpha_0 s_0 + \alpha_1 s_1 = 0 \} \subset P$ for
some $[\alpha_0 : \alpha_1] \in \mathbb{P}^1$. By blowing up a semi-Fano
threefold $P$ the resulting dimension of the kernel $k$ of $\rho$
\[ \dim k = n - 1. \]

The three-form Betti number $b_3(Z^\#)$ of the blown-up
threefold $Z^\#$ becomes
\[ b_3(Z^\#) = b_3(P) + 2 \sum_{i=1}^{n} g(C_i), \]
in terms of the three-form Betti number $b_3(P)$ of the semi-Fano threefold $P$ and the genera $g(C_i)$ of the smooth curve components $C_i$. As all these curves $C_i$ lie in the K3 fiber $S$, the genus $g(C_i)$ is readily computed by the adjunction formula

$$g(C_i) = \frac{1}{2} C_i \cdot C_i + 1 ,$$

with the self-intersections $C_i \cdot C_i$ in $S$.

Locally the singularity looks near $\pi^{-1}([1, 0])$ with
\[ t = \alpha_1 / \alpha_0 \quad \text{like} \]

\[ s_{0,1} \cdots s_{0,n} + ts_1 = 0 . \]

In particular near the intersection locii
\[ \mathcal{I}_{ij} = \{ t = 0 \} \cap \{ s_1 = 0 \} \cap \{ s_{0,i} = 0 \} \cap \{ s_{0,j} = 0 \} \]

it looks like conifolds, which are away from the asymptotic K3 fibres at \( \alpha_0, \alpha_1 \neq 0 \) involved in the gluing. This yields the following abelian gauge group

\[ U(1)^{n-1} \sim \frac{U(1)_1 \times \ldots \times U(1)_n}{U(1)_{\text{diag}}} \]

with the matter whose charges are determined by the
intersections $\chi_{ij} = C_j \cdot C_j$ and as displayed in the table:

<table>
<thead>
<tr>
<th>Multiplicity</th>
<th>$\mathcal{N} = 2$ multiplets $U(1)^{n-1}$ charges</th>
<th>multiplet</th>
<th>$\mathcal{N} = 1$ multiplets $U(1)^{n-1}$ charges</th>
<th>multiplet</th>
</tr>
</thead>
<tbody>
<tr>
<td>$n - 1$</td>
<td>$(0, 0, \ldots, 0)$</td>
<td>vector</td>
<td>$(0, \ldots, 0)$</td>
<td>vector</td>
</tr>
<tr>
<td>$\chi_{ij}$</td>
<td>$(0, \ldots, +1_i, \ldots, +1_j, \ldots, 0)$</td>
<td>hyper</td>
<td>$(0, \ldots, +1_i, \ldots, +1_j, \ldots, 0)$</td>
<td>chiral</td>
</tr>
<tr>
<td>$\chi_{in}$</td>
<td>$(0, \ldots, +1_i, \ldots, 0)$</td>
<td>hyper</td>
<td>$(0, \ldots, +1_i, \ldots, 0)$</td>
<td>chiral</td>
</tr>
</tbody>
</table>

Table 2: The table shows the spectrum of the Abelian $\mathcal{N} = 2$ gauge theory sector arising from the conifold singularities in the building block $(Z_{\text{sing}}, S)$.

Higgs Transitions:
\[
\begin{align*}
    b_2(Y^a) &= b_2(Y^b) - n - 1 \\
    b_3(Y^a) &= b_3(Y^b) + 2 \left( \sum_{1 \leq i < j \leq n} X_{ij} \right) - 3(n - 1)
\end{align*}
\]

2 Non-abelian Gauge symmetry, charged matter spectrum

Let us now turn to the enhancement to non-Abelian \( \mathcal{N} = 2 \) gauge theory sectors in the context of twisted connected \( G_2 \)-manifolds. Let us assume that the anti-canonical line bundle \(-K_P\) of the semi-Fano
threefold $P$ factors as

$$-K_P = \tilde{\mathcal{L}}_1^{\otimes k_1} \otimes \ldots \otimes \tilde{\mathcal{L}}_s^{\otimes k_s} \quad \text{with} \quad n = k_1 + \ldots + k_s,$$

where $\tilde{\mathcal{L}}_i$ are line bundles with global sections $\tilde{s}_{0,i}$. Then the global section $s_0$ of $-K_P$ can further degenerate to $s_0 = \tilde{s}_{0,1}^{k_1} \cdots \tilde{s}_{0,s}^{k_s}$ and the singular building block is of the form

$$Z_{\text{sing}} = \left\{ (x, z) \in P \times \mathbb{P}^1 \left| \alpha_0 \tilde{s}_{0,1}^{k_1} \cdots \tilde{s}_{0,s}^{k_s} + \alpha_1 s_1 = 0 \right. \right\},$$

with the singular equation in the affine coordinate $t = \frac{z_1}{z_0}$. 

(3)
given by

$$\tilde{s}_{0,1}^{k_1} \cdots \tilde{s}_{0,s}^{k_s} + ts_1 = 0.$$ 

As before we assume that all curves $$\tilde{C}_i = \{\tilde{s}_{0,i} = 0\} \cap \{s_1 = 0\}$$ are smooth. In the vicinity of the singular fiber $$\pi^{-1}([1, 0]) \subset Z_{\text{sing}}$$ the singular building block $$(Z_{\text{sing}}, S')$$ develops $$A_{k_i - 1}$$-singularities along those curves $$\tilde{C}_i$$ with $$k_i > 1.$$
<table>
<thead>
<tr>
<th>Multiplicity</th>
<th>$\mathcal{N} = 2$ multiplets</th>
<th>$\mathcal{N} = 1$ multiplets</th>
</tr>
</thead>
<tbody>
<tr>
<td>$s - 1$</td>
<td>$1$</td>
<td>$1$</td>
</tr>
<tr>
<td>$i = 1, \ldots, s$</td>
<td>adj$_{SU(k_i)}$</td>
<td>adj$_{SU(k_i)}$</td>
</tr>
<tr>
<td></td>
<td>$SU(k_i)$ vector</td>
<td>$SU(k_i)$ vector</td>
</tr>
<tr>
<td>$g(\tilde{C_i})$</td>
<td>adj$_{SU(k_i)}$</td>
<td>adj$_{SU(k_i)}$</td>
</tr>
<tr>
<td>$1 \leq i \leq s$</td>
<td>hyper</td>
<td>chiral</td>
</tr>
<tr>
<td>$\tilde{\chi}_{ij}$</td>
<td>$(k_i, k_j)(+1_i,+1_j)$</td>
<td>$(k_i, k_j)(+1_i,+1_j)$</td>
</tr>
<tr>
<td>$1 \leq i &lt; j &lt; s$</td>
<td>hyper</td>
<td>chiral</td>
</tr>
<tr>
<td>$\tilde{\chi}_{in}$</td>
<td>$(k_i, k_n)(+1_i)$</td>
<td>$(k_i, k_n)(+1_i)$</td>
</tr>
<tr>
<td>$1 \leq i &lt; n$</td>
<td>chiral</td>
<td>chiral</td>
</tr>
</tbody>
</table>

**Table 3:** The table shows the spectrum of the $\mathcal{N} = 2$ gauge theory sector with gauge group $G = SU(k_1) \times \ldots \times SU(k_s) \times U(1)^{s-1}$ as arising from the non-Abelian building blocks $(Z_{\text{sing}}, S)$. It lists both the four-dimensional $\mathcal{N} = 2$ and the four-dimensional $\mathcal{N} = 1$ multiplet structure. The adjoint matter is determined by the genus $g(\tilde{C}_i)$ of the curves $\tilde{C}_i$, whereas the bi-fundamental matter from their intersection numbers $\tilde{\chi}_{ij}$ within the K3 surface $S$. 
<table>
<thead>
<tr>
<th>No.</th>
<th>ρ</th>
<th>Gauge Group</th>
<th>( \mathcal{N} = 2 ) Hypermultiplet spectrum</th>
<th>( h^b )</th>
<th>( c^# )</th>
<th>( b^b_3 )</th>
<th>( b^3_3 )</th>
<th>( k^# )</th>
</tr>
</thead>
<tbody>
<tr>
<td>K24, MM34 (_2)</td>
<td>2</td>
<td>( SU(3) \times SU(2) \times U(1) )</td>
<td>(2 \times (\text{adj, 1}); (1, \text{adj}); 3 \times (3, 2)_{+1})</td>
<td>50</td>
<td>14</td>
<td>79</td>
<td>43</td>
<td>4</td>
</tr>
<tr>
<td>K32</td>
<td>2</td>
<td>( SU(3)^2 \times U(1) )</td>
<td>((\text{adj, 1}); (1, \text{adj}); 3 \times (3, 3)_{+1})</td>
<td>52</td>
<td>13</td>
<td>79</td>
<td>40</td>
<td>5</td>
</tr>
<tr>
<td>K35, MM36 (_2)</td>
<td>2</td>
<td>( SU(5) \times SU(2) \times U(1) )</td>
<td>(2 \times (\text{adj, 1}; (5, 2)_{+1})</td>
<td>60</td>
<td>22</td>
<td>87</td>
<td>49</td>
<td>6</td>
</tr>
<tr>
<td>K36, MM35 (_2)</td>
<td>2</td>
<td>( SU(4) \times SU(2) \times U(1) )</td>
<td>(2 \times (\text{adj, 1}); 2 \times (4, 2)_{+1})</td>
<td>54</td>
<td>17</td>
<td>81</td>
<td>44</td>
<td>5</td>
</tr>
<tr>
<td>K37, MM33 (_2)</td>
<td>2</td>
<td>( SU(4) \times SU(3) \times U(1) )</td>
<td>(\text{adj, 1}); 3 \times (4, 3)_{+1})</td>
<td>54</td>
<td>12</td>
<td>79</td>
<td>37</td>
<td>6</td>
</tr>
</tbody>
</table>

**Table 4:** The table exhibits the \( \mathcal{N} = 2 \) gauge theory sectors for some smooth toric semi-Fano threefolds \( P_\Sigma \) of Picard rank two and higher. The columns display the number of the threefold \( P_\Sigma \) in the Mori–Mukai and/or Kasprzyk classification, its Picard rank \( \rho \), the maximally enhanced gauge group of maximal rank by factorizing the anti-canonical bundle, the \( \mathcal{N} = 2 \) matter hypermultiplets, the complex dimensions \( h^b \) and \( c^\# \) of the Higgs and Coulomb branches, the reduced three-form Betti numbers \( b^b_3 \) and \( b^3_3 \), and the kernel \( k^\# \) of the Coulomb branch.
Conclusions

- $10^8$ new $G_2$ manifolds
- universal moduli, splitting of spectrum in sectors
- abelian and non abelian gauge symmetries with non chiral matter
- $G_2$ transitions