Strings on $\text{AdS}_3 \times S^3 \times S^3 \times S^1$

Lorenz Eberhardt

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Based on work with Matthias Gaberdiel, Rajesh Gopakumar and Wei Li

[arXiv:1701.03552], [arXiv:1707.?????]
The background supports the large $\mathcal{N} = 4$ superconformal algebra and is thus very interesting, but still mysterious.

We have analyzed the BPS spectrum of $\text{AdS}_3 \times S^3 \times S^3 \times S^1$ both in string theory and supergravity.

The sugra calculation shows a discrepancy with the old result of [de Boer, Pasquinucci, Skenderis '99].

We have made an explicit proposal for the dual CFT.
The large $\mathcal{N} = 4$ superconformal algebra

- **R-symmetry** [Sevrin, Troost, van Proeyen, Schoutens, Spindel, Theodoridis, Goddard, Schwimmer 88’-90’]:
  - $\mathfrak{su}(2)_{k^+} \oplus \mathfrak{su}(2)_{k^-} \oplus \mathfrak{u}(1)$-current algebra

- **Central charge**:
  \[ c = \frac{6k^+ k^-}{k^+ + k^-} \]

- **Global algebra (wedge-algebra)**: $D(2, 1|\alpha)$. 

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BPS bound

- Representations are labelled by

\[(h ; \ j^+, \ j^-, \ u)\]

- Spin w.r.t. \(A^{+,i}\)
- Spin w.r.t. \(A^{-,i}\)
- Conformal weight
- \(u(1)\)-charge

- BPS bound [Gunaydin, Petersen, Taormina, van Proeyen '89, Petersen, Taormina '90]:

\[
h_{\text{BPS}} = \frac{k^+ j^- + k^- j^+}{k^+ + k^-} + \frac{(j^+ - j^-)^2 + u^2}{k^+ + k^-}
\]

invisible in supergravity \(k^\pm \to \infty\)
The global algebra $D(2, 1|\alpha)$

- $u(1)$-current decouples: Only global $\mathfrak{su}(2) \oplus \mathfrak{su}(2)$-symmetry remains
- Representations are labelled by $(h; j^+, j^-)$

\[ h_{\text{BPS}} = \frac{k^+ j^- + k^- j^+}{k^+ + k^-} \]

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BPS bounds

- The $A_\gamma$ (stringy) BPS-bound is stronger than the $D(2, 1|\alpha)$ (sugra) BPS-bound, equality only for $j^+ = j^-$ and $u = 0$.
- Strange consequence: Sugra BPS states with $j^+ \neq j^-$ have to acquire non-trivial quantum corrections to even satisfy the stringy BPS bound [de Boer, Pasquinucci, Skenderis ’99; Gukov, Martinec, Moore, Strominger ’04].
- According to the analysis of [de Boer, Pasquinucci, Skenderis ’99], sugra contains such BPS states.
Worldsheet analysis

WZW model

For pure NS-NS background, the worldsheet theory of the string can be described by a supersymmetric WZW model based on [Elitzur, Feinerman, Giveon, Tsabar '99]:

$$\mathfrak{sl}(2, \mathbb{R})_k^{(1)} \oplus \mathfrak{su}(2)_{k^+}^{(1)} \oplus \mathfrak{su}(2)_{k^-}^{(1)} \oplus \mathfrak{u}(1)^{(1)}.$$  

Criticality of the string theory requires the total central charge to be 15:

$$k = \frac{k^+ k^-}{k^+ + k^-}.$$

The $\mathfrak{sl}(2, \mathbb{R})$-spin $j$ is identified with the conformal weight of the state in the dual CFT [Elitzur, Feinerman, Giveon, Tsabar '99].

⇒ Can study BPS spectrum
Worldsheet BPS spectrum: unflowed sectors

- In NS-sector: Use the one fermion to lower the $\mathfrak{sl}(2, \mathbb{R})$-spin.

$$j = -\frac{1}{2} + \sqrt{\frac{1}{4} + k \left( \frac{j^+(j^+ + 1)}{k^+} + \frac{j^-(j^- + 1)}{k^-} \right)}.$$ 

- Compare with the BPS bound

$$j \geq \frac{k^+ j^- + k^- j^+}{k^+ + k^-} + \frac{(j^+ - j^-)^2}{k^+ + k^-}.$$ 

- The BPS bound is saturated only for $j^+ = j^-$. 
Complete worldsheet BPS spectrum

- Spectrally flowed (long string) sectors contribute more BPS states.
- Complete BPS spectrum: [LE, Gaberdiel, Gopakumar, Li '17]:

\[
\bigoplus_{j \in \frac{1}{2} \mathbb{Z} \setminus \left( \frac{1}{2} \lfloor k \mathbb{Z} \rfloor \setminus \frac{1}{2} \text{lcm}(k^+,k^-) \mathbb{Z} \right)} \left[ j, j, u = 0 \right]_{S} \otimes \left[ j, j, u = 0 \right]_{S}.
\]

- Taking into account the missing chiral primaries:

\[
\bigoplus_{j \in \frac{1}{2} \mathbb{Z}} \left[ j, j, u = 0 \right]_{S} \otimes \left[ j, j, u = 0 \right]_{S}.
\]

- This should be matched with supergravity, which corresponds to the regime \( k \to \infty \).
Sugra BPS spectrum

- Structure of the result:

\[ h = + \frac{1}{2} + \sqrt{\frac{1}{4} + k \left( \frac{j^+(j^++1)}{k^+} + \frac{j^-(j^-+1)}{k^-} \right)} \]

\[ m^2 = h(h-1) \]

\[ m^2 = \frac{j^+(j^++1)}{k^+} + \frac{j^-(j^-+1)}{k^-} \]

Laplacian on \( S^3 \times S^3 \)

⇒ Looks like a KK-reduction, suggests that same conclusion should also hold true in supergravity.
To fix the sign and to confirm this, we performed an explicit KK-reduction of 9d supergravity on $S^3 \times S^3$.

Result:

- The spectrum arranges itself into $D(2, 1|\alpha)$-multiplets.
- Confirms the string theory result [LE, Gaberdiel, Gopakumar, Li '17]:

\[ j^+ = j^- \]
Comparison with de Boer et al.

- Gives an elegant resolution of the previous puzzle: Sugra has no BPS states for $j^+ \neq j^-$
  \[ \Rightarrow \text{There is no need for miraculous quantum corrections in string theory.} \]
- Previously excluded candidates as dual CFTs are again “back in the game”.

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Matching of the BPS spectrum with the dual CFT

- At generic points in the moduli space and in the large spin limit, the BPS spectrum was independently analysed by [Baggio, Ohlsson Sax, Sfondrini, Stefanski, Torielli ’17] using integrability techniques.

  ⇒ This suggests that the full BPS spectrum is the same everywhere in moduli space, i.e., also the dual CFT should just have BPS states with $j^+ = j^-$. 

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Brane construction

- Wrap a special Lagrangian $S^3$:

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- This configuration gives the near-horizon geometry $\text{AdS}_3 \times S^3 \times S^3 \times S^1$. 

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Worldvolume theory

- The low-energy theory on the 6-dimensional D5-brane worldvolume is a 3-dimensional $U(Q_5^+)$ Chern-Simons theory living in 059.
- Near-horizon limit: overall $U(1)$ decouples.
  $\Rightarrow$ Subtle issue: End up with $SU(Q_5^+)$ or $SU(Q_5^+)/\mathbb{Z}_{Q_5^+}$.
- [Witten '99]: The latter is anomalous unless $Q_5^+ \mid Q_5^-$.  
  $\Rightarrow$ Brane picture is not consistent unless $Q_5^+ \mid Q_5^-$. 
Instanton moduli space

- The dual CFT should be identified with the low-energy theory living on the D1-D5 brane intersection.
- D1-branes can be viewed as instantons in the D5-branes, living on the transverse direction of the D1-branes: $S^3 \times S^1$.
- The dual CFT is the supersymmetric $\sigma$-model on the moduli space $\mathcal{M}_{Q_1, Q_5^+, Q_5^-}$ of $Q_1$ instantons of $SU(Q_5^+)$ on $S^3_{Q_5^- - Q_5^+} \times S^1$. 
Instanton moduli space

- For $Q_5^+ = 1$, the moduli space is easy to determine:

$$\mathcal{M}_{Q_1,1,Q_5^-} \cong \text{Sym}^{Q_1}(S^{3}_{Q_5^-/Q_5^+} - 1 \times S^1).$$

- In general hard, but when $Q_5^+ | Q_5^-$ there is a natural guess:

$$\mathcal{M}_{Q_1,Q_5^+,Q_5^-} \cong \text{Sym}^{Q_1 Q_5^+}(S^{3}_{Q_5^-} - 1 \times S^1).$$

- Supersymmetric $\sigma$-models on these spaces support the large $\mathcal{N} = 4$ algebra with the correct levels.
The symmetric orbifold of $S_\kappa$.

### The theory $S_\kappa$

- $S_\kappa$ is the $\mathcal{N} = 1$ WZW model on $S^3 \times S^1 \cong SU(2) \times U(1)$ [Sevrin, Troost, van Proeyen '88].
- $\kappa$ is the level of the bosonic $\mathfrak{su}(2)$-algebra.
- Fermions generate the current algebra $\mathfrak{su}(2)_1 \oplus \mathfrak{su}(2)_1$.
- Theory supports the $A_\gamma$ algebra with levels $k^+ = 1$, $k^- = \kappa + 1$. 

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Suresh Yenduri

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The symmetric orbifold of $S_\kappa$

- Moduli spaces were of the form $\text{Sym}^N(S_\kappa)$.
- This supports the large $\mathcal{N} = 4$ algebra with levels $(N, N(\kappa + 1))$.
- The same theories were considered before in [Elitzur, Feinerman, Giveon, Tsabar '98; Gukov, Martinec, Moore, Strominger '04], but discarded because of the wrong BPS spectrum.
The BPS spectrum of the symmetric orbifold of $S_\kappa$ and comparison

- Complete low-lying BPS spectrum:
  \[ \frac{c}{12} \bigoplus_{j=0}^\infty [j, j, u = 0]_S \otimes [j, j, u = 0]_S. \]

- Perfect agreement with the string theory prediction!
Conclusions

- We have shown that the BPS spectrum of string theory and sugra on $\text{AdS}_3 \times S^3 \times S^3 \times S^1$ agrees and contains only states with $j^+ = j^-$.  
- We analyzed the BPS spectrum of the symmetric product of the theory $S_\kappa$ and found precisely the same BPS spectrum.  
- Very convincing evidence in favour of the duality!
Thank you!