Phases of Gauge Theories and Surface Operators

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based on:

• S.G., E.Witten, “Rigid Surface Operators,” arXiv:0804.1561
• S.G., E.Witten, “Gauge theory, ramification, and the geometric Langlands program,” hep-th/0612073
• work in progress with N.Seiberg
Phase Diagram of Water

The phase diagram of water illustrates the different phases of water (solid, liquid, and gas) as functions of temperature and pressure. Key points on the diagram include:

- **Solid** phase region
- **Liquid** phase region
- **Gas** phase region
- **Critical point**
- **Supercritical fluid** region
- **Normal boiling point**
- **Normal melting point**
- **Triple point**

The diagram shows the transition points and the regions where different phases exist under specified conditions of temperature and pressure.
Phase Diagram of QCD

*M. Stephanov*
Phases of $N=1$ Gauge Theories

- Moduli space of $N=1$ SYM with an adjoint matter $\Phi$ and a superpotential $W(\Phi)$

- Phases not distinguished by traditional order parameters, like Wilson and 't Hooft operators

[F. Cachazo, N. Seiberg, E. Witten]
Operators in 4D Gauge Theory

- **Codimension 4**: Local operators
  much studied in AdS/CFT

- **Codimension 3**: Line operators:
  - Wilson line
  - 't Hooft line

- **Codimension 2**: Surface operators

- **Codimension 1**: Boundaries
Wilson Operators

\[ W_R (\gamma) = \text{Tr}_R \ 	ext{Hol}_{\gamma} (A) = \text{Tr}_R (P \exp \gamma A) \]

representation of the gauge group \( G \)
Wilson Operators

in abelian gauge theory:

\[ W_q (\gamma) = \exp \left( i q \oint_D A \right) \]

\[ = \exp \left( i q \int_D F \right) \]

“surface operator”
**Wilson Operators**

\[ T \gg L \gg 0 \]

\[ \langle W_q(y) \rangle = \text{const.} \ e^{-TV(L)} \]

\[ V(L) = \text{interaction energy of the charges at distance } L \]

(Quark-antiquark potential)
Phases

- Coulomb:
  \[ V(L) \sim \frac{1}{L} \sim \langle W(\delta) \rangle \sim e^{-\frac{T}{L}} \]

- Higgs:
  \[ V(L) \sim \text{const} \sim \langle W(\delta) \rangle \sim e^{-\frac{T}{L}} \]

- confinement:
  \[ V(L) \sim L \sim \langle W(\delta) \rangle \sim e^{-s_0} \]
't Hooft Operators

• remove $\gamma$ from $M$ \(\sim\) $M \setminus \gamma$ has $S^2$

\[ T_q(\gamma) : \quad \int_{S^2} F = 2\pi q \]

• For general $G$: $T_p(\gamma)$

\[ p : U(1) \rightarrow G \quad \text{cf.} \quad p : SU(2) \rightarrow G \]

for surface operators
't Hooft Operators Detect Spontaneous Symmetry Breaking

- in Abelian Higgs model:

\[ L(A, \phi) = \frac{1}{2e^2} F_A^2 + \frac{1}{2e^2} |D_A|^2 + \frac{\lambda}{4} (|\phi|^2 - v^2)^2 \]

\[ v^2 > 0: \quad \text{Higgs phase, mass gap} \quad \text{vortices} \]

\[ G = U(1) \rightarrow 1 \]
't Hooft Operators Detect Spontaneous Symmetry Breaking

vortex:

\[ \int \frac{F}{2\pi} = 1 \]

suppose it has a boundary:

\[ \int_{S^2} \frac{F}{2\pi} = 1 \]
‘t Hooft Operators Detect Spontaneous Symmetry Breaking

\[ T_{q=1}(\gamma) = \text{boundary of a vortex supported on a surface } D, \text{ s.t. } \partial D = \gamma \]

\[ \langle T_{q=1}(\gamma) \rangle \sim e^{-c \cdot \text{Area}(D)} \]

in Higgs phase
Electric-Magnetic Duality

\[ G \leftrightarrow \quad \gamma G \]

\[ \tau \leftrightarrow -\frac{1}{\tau} \]

\[ R \leftrightarrow \quad \xi R \]

\[ W_{\tau R}(\delta) \leftrightarrow \quad T_{\tau R}(\delta) \]
Electric-Magnetic Duality

Confinement (mass gap) \[\langle w \rangle \sim e^{-\text{Area}}\] \[\langle T \rangle \sim e^{-\text{Length}}\]

Higgs (mass gap) \[\langle T \rangle \sim e^{-\text{Area}}\] \[\langle w \rangle \sim e^{-\text{Length}}\]

Coulomb (no mass gap)
Surface Operators

• supported on a surface $D$ in a space-time manifold $M$

• defined by introducing a singularity for the gauge field (for simplicity, take $G=U(1)$):

$$ F = 2\pi i \alpha \delta_D $$

• and a phase factor in the path integral:

$$ \exp \left( i\eta \int_D F \right) $$
Surface Operators

Suppose $\mathcal{D}$ has a boundary:

\[
\int \frac{F}{2\pi} = \alpha
\]

\[
\int_{S^2} \frac{F}{2\pi} = \alpha
\]

$\alpha = \text{magnetic charge}$
Surface Operators

A surface operator with parameters \((\alpha, \eta)\) can be thought of as a Dirac string of a dyon with magnetic charge \(\alpha\) and electric charge \(\eta\).

One might expect that surface operators are labeled by representations of the gauge group \(G\) (or the dual group \(\hat{G}\)), just like electric and magnetic charges.
Indeed, for $G = U(N)$, there are different types of surface operators labeled by partitions of $N$:

\[ N = 3 + 3 + 2 + 2 + 1 \]

- analog for general $G$:

\[ \rho : SU(2) \rightarrow G \]

- in $SO(N)$ and $Sp(N)$ gauge theory, correspond to partitions of $N$ with certain constraints
**Surface operators shown in red and labeled by * appear to spoil S-duality. In order to restore a nice match, one has to introduce a larger class of surface operators.**
Holographic Dual

- In the limit of large $N$ and large 't Hooft coupling, such surface operators can be described as D3-branes in $\text{AdS}_5 \times S^5$ with world-volume $Q \times S^1$ where $S^1 \subset S^5$ and $Q \subset \text{AdS}_5$ is a volume minimizing 3-manifold with boundary $\partial Q = D \subset M$.
• Surface operators exhibit a “volume law” when theory admits *domain walls*, which can end on a surface operator:

$$\oint A = 2\pi \alpha$$

• Examples of such theories include *N=1* Dijkgraaf-Vafa type theories.
Thermal Phase Transition

- To study thermal phase transition in N=4 SYM theory, we compactify the time direction on a circle of circumference $\beta = 2\pi/T$ and study the theory on a space-time manifold $M = S^1_\beta \times S^3$ with thermal (anti-periodic) boundary conditions on fermions.

- It is dual to IIB string theory on $X \times S^5$ where $C$

\[
X = \begin{ cases}
\text{thermal AdS} & \text{(low temperature)} \\
B^4 \times S^1 \\
\text{AdS black hole} & \text{(high temperature)} \\
S^3 \times B^2
\end{ cases}
\]
Low Temperature

- temporal surface operator ($D = \gamma \times S_{\beta}^1$):

$$\langle O_{\text{temporal}} \rangle = 0$$

since $S_{\beta}^1$ is not contractible in $X$, and so there is no minimal submanifold $Q$ bounded by $D$.

- spatial surface operator ($D \subset S^3$):

$$\langle O_D \rangle = e^{-\text{Area}(D)}$$
High Temperature

- temporal surface operator ($D = \gamma \times S^1$):
  $$\langle O_{\text{temporal}} \rangle \neq 0$$

- spatial surface operator ($D \subset S^3$):
  $$\langle O_D \rangle = e^{-\text{Volume}(D)}$$

the warp factor is bounded below
From Surfaces to Lines

• Note, in the high temperature limit ($\beta \to 0$) the theory reduces to a pure (non-supersymmetric) three-dimensional Yang-Mills theory on $S^3$. (Scalars acquire a mass from loops.)

• In this limit, a temporal surface operator turns into a line operator (supported on $\gamma$) in the 3D theory.

• Therefore, surface operators in the four-dimensional gauge theory exhibit volume (resp. area) law whenever the corresponding line operators in the 3D theory exhibit area (resp. circumference) law.