Amplitudes in N=8 supergravity and Wilson Loops

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Brandhuber, Heslop, GT — 0707.1153 [hep-th]
Brandhuber, Heslop, Nasti, Spence, GT — 0805.2763 [hep-th]

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Outline

- MHV amplitudes in N=4 SYM & Wilson loops
  - iterative structures in the perturbative expansion
  - one-loop n-point amplitudes and Wilson loops

    (Brandhuber, Heslop, GT)
- 4-point MHV amplitude in N=8 supergravity

  (Brandhuber, Heslop, Nasti, Spence, GT)
  - iterative structures
  - Wilson loops

Motivations

- Scattering amplitudes in gauge theory are simple
  - geometry in Twistor Space (Witten)
  - recursive structures in the perturbative S-matrix of gauge theories
- Simplicity hidden by Feynman diagrams
  - diagrams not not separately gauge invariant
  - unphysical singularities
- Unitarity-based & twistor-inspired methods
  - gauge-invariant, on-shell data at each intermediate step of calculation
  - also in non-supersymmetric theories

- Amplitudes in N=4 super Yang-Mills are even simpler (and more mysterious...)

  - All one-loop amplitudes expressed in terms of box functions (Bern, Dixon, Dunbar, Kosower)

  - Iterative structures in splitting amplitudes and planar MHV amplitudes (Anastasiou, Bern, Dixon, Kosower; Bern, Dixon, Smirnov)
    Lance Dixon's talk

    - planar: leading in $1/N$

- Intriguing connection between MHV amplitudes in N=4 super Yang-Mills and Wilson loops
  (Drummond, Korchemsky, Sokatchev + Henn; Brandhuber, Heslop, GT)

- Dual (super)conformal symmetry:
  Emery Sokatchev & Gregory Korchemsky talks tomorrow

  - integral functions in planar amplitudes are pseudo-conformal
    (Drummond, Henn, Smirnov, Sokatchev)

  - Wilson loops satisfy dual conformal Ward identities
    (Drummond, Henn, Korchemsky, Sokatchev)

- Maximal transcendentality

- Novel motivation: explore the new duality N=4 MHV amplitudes/Wilson loops in other theories
  - Wilson loop calculation does not produce spinor prefactors
  - Look at amplitudes which are proportional to the tree-level amplitude to all loops, $\mathcal{A} = \mathcal{A}_{\rm tree}\,\mathcal{M}$ ...
    - ... where $\mathcal{M}$ is a scalar, helicity-blind function
    - Q: can we calculate it using Wilson loops ?
  - First, we need to find some examples ...

- We will consider N=8 supergravity amplitudes
  - four-point MHV amplitude is of the form $\mathcal{A} = \mathcal{A}_{\rm tree}\,\mathcal{M}$
  - maximally supersymmetric
  - nonplanar
- Our goals:
  - look for iterative relations in MHV amplitudes
  - try to relate amplitudes to Wilson loops
  - idea: find more similarities between the two maximally supersymmetric theories

- Common features N=4/N=8:
  - Absence of triangle and bubble subgraphs in amplitudes ("no-triangle hypothesis") (Bern, Dixon, Perelstein, Rozowsky; Bern, Bjerrum-Bohr, Dunbar; Bjerrum-Bohr, Dunbar, Ita; Bjerrum-Bohr, Dunbar, Ita, Perkins, Risager; Bjerrum-Bohr, Vanhove)
  - N=8 conjectured to be perturbatively finite (Bjerrum-Bohr, Dunbar, Ita, Perkins, Risager; Chalmers; Bern, Dixon, Roiban; Green, Russo, Vanhove; Bern, Carrasco, Dixon, Johansson, Kosower, Roiban) Zvi Bern and David Dunbar talks today
- Gauge theory/gravity:
  - KLT relations (Kawai, Lewellen, Tye)
  - UV behaviour of tree amplitudes under (complex) shifts much softer than expected, tree-level recursion relations (Bedford, Brandhuber, Spence, GT; Cachazo, Svrcek; Benincasa, Boucher-Veronneau, Cachazo; Arkany-Hamed, Kaplan)

In the rest of the talk:

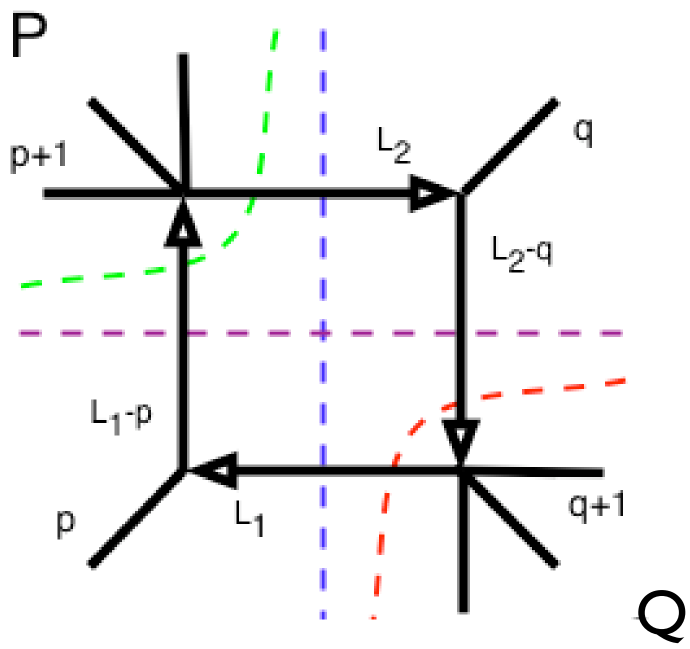
- Iterative structures
  - n-point MHV amplitudes in N=4 SYM
    (Anastasiou, Bern, Dixon, Kosower; Bern, Dixon, Smirnov)
  - 4-point MHV amplitude in N=8 supergravity
    (Brandhuber, Heslop, Nasti, Spence, GT)
- Wilson loops
  - vs one-loop n-point MHV amplitudes in N=4 SYM
    (Brandhuber, Heslop, GT)
  - vs one-loop 4-point MHV amplitude in N=8 supergravity
    (Brandhuber, Heslop, Nasti, Spence, GT)

N=4 Yang-Mills

Simplest one-loop amplitude

- n-point MHV amplitude in N=4 SYM at one loop:

$$\mathcal{A}_{\mathrm{MHV}}^{1-\mathrm{loop}} = \mathcal{A}_{\mathrm{MHV}}^{\mathrm{tree}} \sum$$



- Colour-ordered partial amplitude, leading term in $1/N$
- Sum of two-mass easy box functions, all with coefficient 1

Diagrammatic interpretation

- Computed in 1994 by Bern, Dixon, Dunbar and Kosower using unitarity
- Rederived in 2004 with loop MHV diagrams...

  (Brandhuber, Spence, GT)
- ...and, more recently, with a weakly-coupled Wilson loop calculation, with the Alday-Maldacena polygonal contour (Brandhuber, Heslop, GT)

Surprising regularities at higher loops

(Anastasiou, Bern, Dixon, Kosower; Bern, Dixon, Smirnov)

- n-point MHV amplitude in N=4 SYM

  - $\mathcal{A}_{n,\mathrm{MHV}} = \mathcal{A}_{n,\mathrm{MHV}}^{\mathrm{tree}} \mathcal{M}_n$

$$\mathcal{M}_n := 1 + \sum_{L=1}^{\infty} a^L \mathcal{M}_n^{(L)} \stackrel{?}{=} \exp\left[\sum_{L=1}^{\infty} a^L \left(f^{(L)}(\epsilon)\mathcal{M}_n^{(1)}(L\epsilon) + C^{(L)} + \mathcal{O}(\epsilon)\right)\right]$$

$a \sim g^2 N/(8\pi^2)$ (Bern, Dixon, Smirnov)

  - $\mathcal{M}_n^{(1)}(\epsilon)$ is the all-orders in ϵ one-loop amplitude, $D = 4 - 2\epsilon$

  - $f^{(L)}(\epsilon) = f_0^{(L)} + \epsilon f_1^{(L)} + \epsilon^2 f_2^{(L)}$

☝ anomalous dimension of twist-two operators at large spin, $\gamma_K^{(L)}/4$

☞ Higher-loop amplitudes expressed in terms of lower loop amplitudes

First few terms of BDS conjecture:

(take Log of the Ansatz)

$$\mathcal{M}_n^{(2)} = \tfrac{1}{2}\Big(\mathcal{M}_n^{(1)}(\epsilon)\Big)^2 + f^{(2)}(\epsilon)\,\mathcal{M}_n^{(1)}(2\epsilon) + \mathcal{O}(\epsilon)$$

$$\mathcal{M}_n^{(3)} = -\tfrac{1}{3}\Big(\mathcal{M}_n^{(1)}(\epsilon)\Big)^3 + \mathcal{M}_n^{(2)}(\epsilon)\mathcal{M}_n^{(1)}(\epsilon) + f^{(3)}(\epsilon)\,\mathcal{M}_n^{(1)}(3\epsilon) + \mathcal{O}(\epsilon)$$

and so on...

- Signature of two-loop iteration:

$$\mathcal{M}_n^{(2)} - \tfrac{1}{2}\Big(\mathcal{M}_n^{(1)}(\epsilon)\Big)^2 = f^{(2)}(\epsilon)\,\mathcal{M}_n^{(1)}(2\epsilon) + \mathcal{O}(\epsilon)$$

One-loop amplitude

  - Requires knowledge of lower-loop amplitude to higher orders in ϵ
  - Go up by one loop only

IR behaviour of Yang-Mills amplitudes

- Motivates BDS Ansatz (Anastasiou, Bern, Dixon, Kosower)
- Universal resummation of IR divergences

$$\mathcal{A}|_{\mathrm{IR}} = \prod_{i=1}^{n} \mathcal{A}_{\mathrm{div}}(s_{ij})$$

(for colour-ordered amplitudes)

$$\mathcal{A}_{\mathrm{div}}(s) \;=\; \exp\left[-\frac{1}{8\epsilon^2}\sum_{L=1}^{\infty} a^L \left(\frac{-s}{\mu^2}\right)^{-L\epsilon} \frac{\gamma_K^{(L)}}{L^2} - \frac{1}{4\epsilon}\sum_{L=1}^{\infty} a^L \left(\frac{-s}{\mu^2}\right)^{-L\epsilon} \frac{g^{(L)}}{L}\right]$$

(Catani; Magnea, Sterman; Sterman, Tejeda-Yeomans)

- BDS: exponentiation of finite parts
  - Exponentiated finite remainders approach constants (independent of kinematics and # of particles)

Checks of BDS conjecture

- Two and three loops at four points (Anastasiou, Bern, Dixon, Kosower; Bern, Dixon, Smirnov). Confirmed result for three-loop cusp anomalous dimension obtained assuming maximal transcendentality (Kotikov, Lipatov, Onishcenko, Velizhanin)

- Two loops at five points (Bern, Czakon, Kosower, Roiban, Smirnov)

  - Parity odd terms cancel in the iteration

- Problems begin at six points (Bern, Dixon, Kosower, Roiban, Spradlin, Vergu, Volovich) Several talks at this workshop

- Exponent requires an additional finite remainder

N=8 Supergravity

N=8 supergravity MHV amplitudes

- At four points $\quad \mathcal{A}_{4,\mathrm{MHV}}^{\mathcal{N}=8} = \mathcal{A}_{4,\mathrm{MHV}}^{\mathrm{tree}} \mathcal{M}_4^{\mathcal{N}=8}$
  - tree-level amplitude factors out as in N=4 thanks to supersymmetric Ward identities

- Write $\quad \mathcal{M}_4^{\mathcal{N}=8} = 1 + \sum_{L=1}^{\infty} \mathcal{M}_4^{(L)} = \exp\left[\sum_{L=1}^{\infty} m_4^{(L)}\right]$

$$m_4^{(1)} = \mathcal{M}_4^{(1)}\ , \quad m_4^{(2)} = \mathcal{M}_4^{(2)} - \frac{1}{2}\left(\mathcal{M}_4^{(1)}\right)^2$$

and so on

- Goal: compute the quantity $\quad \mathcal{M}_n^{(2)} - \frac{1}{2}\left(\mathcal{M}_n^{(1)}(\epsilon)\right)^2$

In YM: $\quad \mathcal{M}_{\mathrm{YM}}^{(2)} - \frac{1}{2}\left(\mathcal{M}_{\mathrm{YM}}^{(1)}(\epsilon)\right)^2 = f^{(2)}(\epsilon)\, \mathcal{M}_{\mathrm{YM}}^{(1)}(2\epsilon) + \mathcal{O}(\epsilon)$

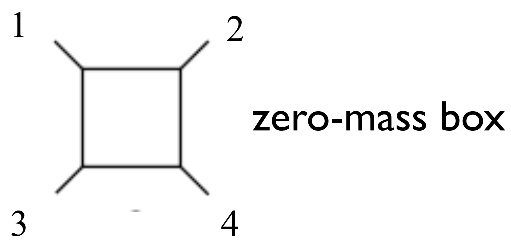
One- and two-loop MHV amplitude

- One loop:

$$\mathcal{M}_4^{(1)} = -i\, s\, t\, u \left(\frac{\kappa}{2}\right)^2 \left[\mathcal{I}_4^{(1)}(s,t) + \mathcal{I}_4^{(1)}(s,u) + \mathcal{I}_4^{(1)}(u,t)\right]$$

(Green, Schwarz, Brink; Dunbar, Norridge)

$$\mathcal{I}_4^{(1)}(s,t) := \int \frac{d^D l}{(2\pi)^D} \frac{1}{l^2 (l-p_1)^2 (l-p_1-p_2)^2 (l+p_4)^2}$$



- No colour ordering for gravity
  - sum over permutations (1234), (1342), (1423)

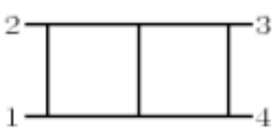
- Two loops:

$$\mathcal{M}_4^{(2)} = \left(\frac{\kappa}{2}\right)^4 stu \left[s^2 \mathcal{I}_4^{(2),\mathrm{P}}(s,t) + s^2 \mathcal{I}_4^{(2),\mathrm{P}}(s,u) + s^2 \mathcal{I}_4^{(2),\mathrm{NP}}(s,t) + s^2 \mathcal{I}_4^{(2),\mathrm{NP}}(s,u) \; + \; \text{cyclic}\right]$$

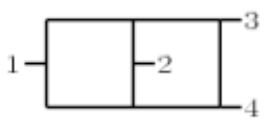
(Bern, Dunbar, Dixon, Perelstein, Rozowsky)

- $\mathcal{I}_4^{(2),\mathrm{P}}, \mathcal{I}_4^{(2),\mathrm{NP}}$ are the planar and non-planar boxes

$$\mathcal{I}_4^{(2),\mathrm{P}}(s,t) = \int \frac{d^D l}{(2\pi)^D} \frac{d^D k}{(2\pi)^D} \frac{1}{l^2\,(l-p_1)^2\,(l-p_1-p_2)^2\,(l+k)^2 k^2\,(k-p_4)^2\,(k-p_3-p_4)^2}$$



$$\mathcal{I}_4^{(2),\mathrm{NP}}(s,t) = \int \frac{d^D l}{(2\pi)^D} \frac{d^D k}{(2\pi)^D} \frac{1}{l^2\,(l-p_2)^2\,(l+k)^2\,(l+k+p_1)^2\,k^2\,(k-p_3)^2\,(k-p_3-p_4)^2}$$



$$s := (p_1+p_2)^2,\; t := (p_2+p_3)^2,\; u := (p_1+p_3)^2$$

- Laurent expansion explicitly evaluated by Smirnov and Tausk
- use it to study possible iterations

Iterative structure

- **Main result:** $\mathcal{M}_n^{(2)} - \frac{1}{2}\left(\mathcal{M}_n^{(1)}(\epsilon)\right)^2 = \text{finite} + \mathcal{O}(\epsilon)$

- Finite remainder has uniform transcendentality
  - $\pi, \log$ have transcendentality 1; ζ_n, Li_n have transcendentality n ...
  - Soft anomalous dimensions in N=4 obtained as leading transcendentality contribution of QCD result (Kotikov, Lipatov, Onishcenko, Velizhanin)

- Transcendentality appears after sum over perm's
  - Planar one- and two-loop box are transcendental; specific combination of nonplanar double-boxes is transcendental

☞ Q: What about higher loops? Is transcendentality a property of N=8 theory ?

- Remainder is "simpler" compared to full $\mathcal{M}_4^{(2)}$

$$\mathcal{M}_4^{(2)} - \frac{1}{2}(\mathcal{M}_4^{(1)})^2 \ = - \left(\frac{\kappa}{8\pi}\right)^4 \Big[u^2 \big[k(y) + k(1/y)\big] \ + \ s^2 \big[k(1-y) + k(1/(1-y)\big] \\ + \ t^2 \big[k(y/(y-1)) + k(1-1/y)\big]\Big] + \mathcal{O}(\epsilon)$$

where

$$\begin{aligned} k(y) \quad &:= \quad \frac{L^4}{6} + \frac{\pi^2 L^2}{2} - 4S_{1,2}(y)L + \frac{1}{6}\log^4(1-y) + 4\ S_{2,2}(y) - \frac{19\pi^4}{90} \\ &+i \quad \left(-\frac{2}{3}\pi\log^3(1-y) - \frac{4}{3}\pi^3\log(1-y) - 4L\pi\ \mathrm{Li}_2(y) + 4\pi\mathrm{Li}_3(y) - 4\pi\zeta(3)\right) \end{aligned}$$

$$y = -s/t\ , \quad L := \log(s/t)$$

Disclaimer: analytic continuations needed !

What about IR divergences ?

- Result for amplitudes should agree with the expected IR divergence structure

- IR behaviour of gravity amplitudes studied by Weinberg in 1965 !

  - Not in dimensional regularisation...

- Much simpler than Yang-Mills amplitudes

Steven Weinberg, Phys. Rev. 140: B516 (1965)

6. Remark

It was crucial in the above that the infrared divergences arise only from diagrams in which the soft real or virtual photon or graviton is attached to an external line, with "external line" *not* including the soft real photons or gravitons themselves. In electrodynamics this is true because photons are electrically neutral. In gravitation theory it is justified because the effective coupling constant for emission of a very soft graviton from a graviton (or photon) line with energy E is proportional to E, and the vanishing of this factor prevents simultaneous infrared divergences from a graviton and the line to which it is attached.

But these remarks do not apply to theories involving charged massless particles. In such theories (including the Yang-Mills theory) a soft photon emitted from an external line can itself emit a pair of soft charged massless particles, which themselves emit soft photons, and so on, building up a cascade of soft massless particles each of which contributes an infrared divergence. The elimination of such complicated interlocking infrared divergences would certainly be a Herculean task, and might even not be possible.

We may be thankful that the zero charge of soft photons and the zero gravitational mass of soft gravitons saves the real world from this mess. Perhaps it would not be too much to suggest that it is the infrared divergences that prohibit the existence of Yang-Mills quanta or other charged massless particles. See Sec. III for further remarks in this direction.

IR behaviour of (super)gravity amplitudes

- Exponentiation of one-loop divergences (Weinberg)
  - Similar to QED
  - Soft and collinear amplitudes unrenormalised

    (Bern, Dunbar, Dixon, Perelstein, Rozowsky)

- No colour ordering: $$\mathcal{M}|_{\rm IR} = \prod_{i<j} \mathcal{M}_{\rm div}(s_{ij})$$

- 4 pts, one loop, $$\mathcal{M}^{(1)}\Big|_{\rm IR} = c_\Gamma \left(\frac{\kappa}{2}\right)^2 \frac{2}{\epsilon} \Big[s\,\log(-s) + t\,\log(-t) + u\,\log(-u)\Big]$$
  - ϵ^{-1} IR divergence, softer than in YM

- Our result is in agreement with the expected IR singularities
  - Cancellation of leading and subleading singularities in the difference $\mathcal{M}_4^{(2)} - \frac{1}{2}(\mathcal{M}_4^{(1)})^2$
  - No new divergent contribution introduced at two loops
- Agreement with the results of Naculich, Nastase, Schnitzer; Dixon (unpublished)

Beyond four points

- One-loop amplitudes no longer proportional to the tree-level amplitude
- Requires more thinking / more ☕ ...

Wilson Loops

Amplitudes in N=4 and Wilson Loops

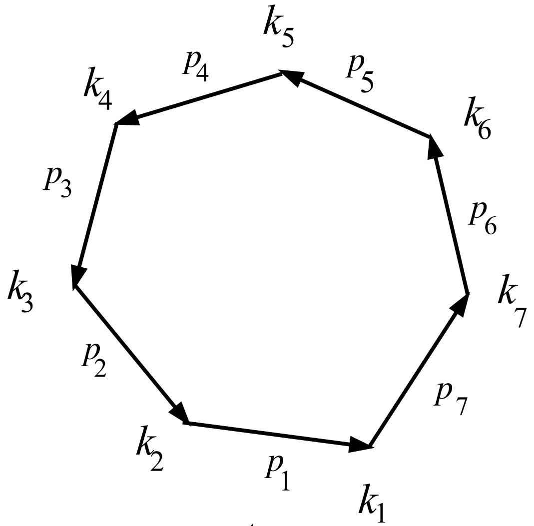
(Drummond, Korchemsky, Sokatchev; Brandhuber, Heslop, GT; Drummond, Henn, Korchemsky, Sokatchev)

- MHV amplitudes in N=4 super Yang-Mills appear in a completely different calculation:

$$< W[C] >$$

- Contour C is determined by the momenta of the scattered particles
- Strong coupling calculation of Alday and Maldacena

- The contour of the Wilson loop:
  - this contour corresponds to a seven-point amplitude
  - colour ordering $\mathrm{Tr}(T^1 T^2 \cdots T^7)$
  - at strong coupling, boundary of worldsheet tends to boundary of dual AdS space as IR cutoff is removed



$p_i = k_i - k_{i+1}$ lightlike momenta

k's are T-dual (region) momenta

  - momentum conservation $\sum_{i=1}^{n} p_i = 0$ ➡ closed contour
  - dual conformal symmetry acts on the T-dual momenta

- Result: $< W[C] >$ is the n-point MHV amplitude in N=4 SYM (modulo tree-level prefactor)

- Unexpected !

  - Eikonal approximation usually only reproduces IR behaviour; we also get finite parts

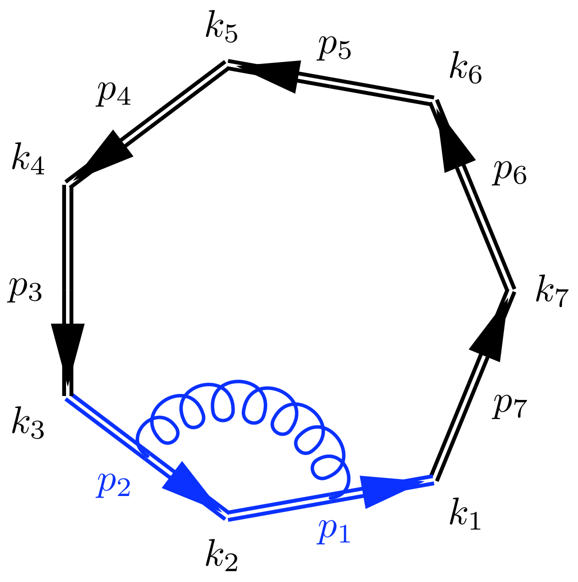
- Conjecture: (Log) $< W[C] >$ = (Log) $\mathcal{M}$ persists at higher loops

  - Recently checked at two loops by Drummond, Henn, Korchemsky, Sokatchev for the four-, five-, and six-point case
  - Gregory Korchemsky's talk tomorrow

$< W[C] >$ at one loop, n points

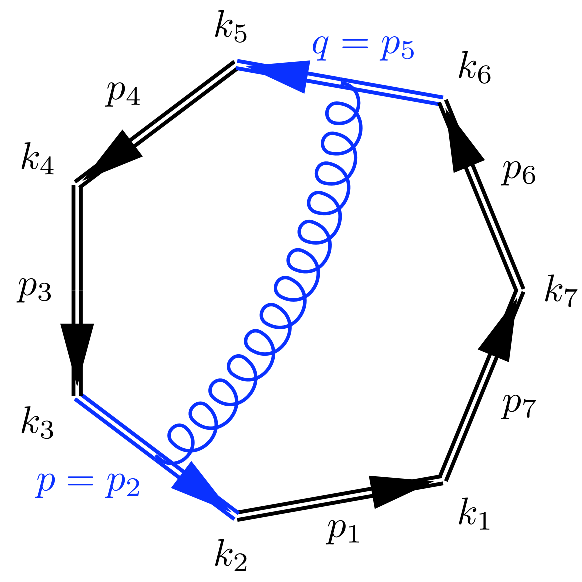
(Brandhuber, Heslop, GT)

- Calculation done (almost) instantly.
  Two classes of diagrams:



Gluon stretched between two segments meeting at a cusp

A. Infrared divergent



Gluon stretched between two non-adjacent segments

B. Infrared finite

- Clean separation between infrared-divergent and infrared-finite terms
    - Important advantage, as ε can be set to zero in the finite parts from the start
- From diagrams in class A :

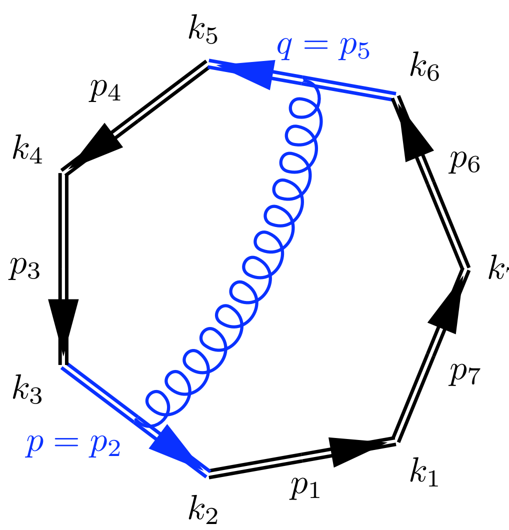
$$\mathcal{M}_n^{(1)}\big|_{IR} = -\frac{1}{\varepsilon^2}\sum_{i=1}^{n}\left(\frac{-s_{i,i+1}}{\mu^2}\right)^{-\varepsilon}$$

    - $s_{i,i+1} = (p_i + p_{i+1})^2$ is the invariant formed with the momenta meeting at the cusp

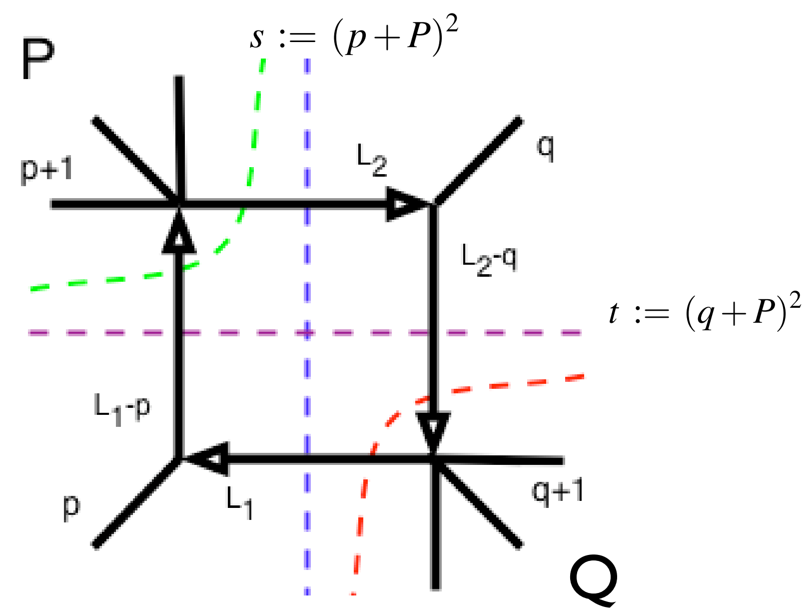
- Diagram in class B, with gluon stretched between p and q gives a result proportional to

$$\mathcal{F}_\varepsilon(s,t,P,Q) = \int_0^1 d\tau_p d\tau_q \, \frac{P^2+Q^2-s-t}{\left[-\left(P^2+(s-P^2)\tau_p+(t-P^2)\tau_q+(-s-t+P^2+Q^2)\tau_p\tau_q\right)\right]^{1+\varepsilon}}$$

- Explicit evaluation shows that this is the finite part of a 2-mass easy box function
  - Two-dimensional representation of a four-dimensional integral function



$\leftrightarrow$



- In the example: $p = p_2 \quad q = p_5$

  $P = p_3 + p_4\,, \quad Q = p_6 + p_7 + p_1$

- One-to-one correspondence between Wilson loop diagrams and finite parts of 2-mass easy box functions

- Explains why each box function appears with coefficient equal to one in the expression of the one-loop N=4 MHV

“Conformal” gauge

(Brandhuber, Heslop, Nasti, Spence, GT)

- A gauge where cusp diagrams vanish
- Motivation: reduce # of diagrams
  - Wilson loop is gauge invariant
  - special case of a Feynman-’t Hooft gauge

$$\mathcal{L}^{(\mathrm{gf})} = \frac{\alpha}{2}\,(\partial_\mu A^\mu)^2$$

- α-gauge fixing
- $\alpha = 1$ usual Feynman gauge
- $\alpha = \frac{\epsilon}{2-\epsilon}$ conformal gauge

- Propagator is $\Delta^{\rm conf}_{\mu\nu}(x) \ \sim \ \frac{\epsilon+1}{\epsilon}\frac{1}{(-x^2+i\varepsilon)^{1+\epsilon}}\left[\eta_{\mu\nu} \ - \ 2\frac{x_\mu x_\nu}{x^2}\right]$

$J_{\mu\nu}(x) := \eta_{\mu\nu} \ - \ 2\frac{x_\mu x_\nu}{x^2}$ Inversion tensor

- Full box function from a single diagram !

Gravity Wilson Loops

(Brandhuber, Heslop, Nasti, Spence, GT)

- Requirements for candidate Wilson loop:
  - invariance under coordinate transformations
  - contour dictated by particle momenta
  - has the same symmetries as the scattering amplitude

Should agree with amplitude !

- Obvious choice: $\langle \mathrm{Tr}\, \mathcal{U}(C) \rangle$ where

$$\mathcal{U}^{\alpha}_{\beta}(C) \ := \ \mathcal{P} \exp \left[i\kappa \oint_C dy^{\mu} \Gamma^{\alpha}_{\mu\beta}(y) \right]$$

  - Γ is the Christoffel connection
  - invariant under coordinate transformations
  - already studied in perturbation theory (Modanese)

- Result has nothing to do with amplitude !

$$\kappa^2 \oint_C dx^{\mu} dy^{\nu} \langle \Gamma^{\alpha}_{\mu\beta}(x) \Gamma^{\beta}_{\nu\alpha}(y) \rangle \ \sim \ \kappa^2 \oint_C \ dx_{\mu} dy^{\mu} \delta^{(D)}(x-y)$$

  - (quadratically) divergent expression, reminiscent of the loop equation...

- Try again
  - work in linearised approximation $g_{\mu\nu}(x) = \eta_{\mu\nu} + \kappa h_{\mu\nu}(x)$

$$W[C] \;:=\; \left\langle \exp\left[i\kappa \oint_C d\tau \; h_{\mu\nu}(x(\tau))\dot{x}^{\mu}(\tau)\dot{x}^{\nu}(\tau) \right] \right\rangle$$

  - Same expression used in gravity eikonal approximation
    (Kabat & Ortiz; Fabbrichesi, Pettorino, Veneziano, Vilkovisky)

- For cusped contours, gauge invariance violated at the cusps
  - Exponent can be rewritten as $\int d^D x \, \mathcal{T}^{\mu\nu}(x) h_{\mu\nu}(x)$

$$\mathcal{T}^{\mu\nu}(x) \; := \; \int d\tau \; \dot{x}^{\mu}(\tau)\dot{x}^{\nu}(\tau) \, \delta^{(D)}(x - x(\tau))$$

energy-momentum tensor of free particle

- Reparametrisation invariance & cusps

- Try anyway
  - in order to have correct symmetries, we consider

$$W \ := \ W[C_{1234}]\,W[C_{1243}]\,W[C_{1324}]$$

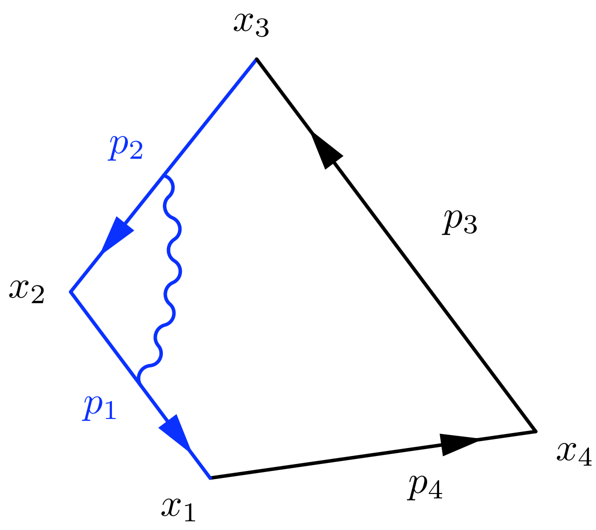
  - C_{ijkl} is a contour obtained by joining p_i, p_j, p_k, p_l in this order
  - At one loop, $W^{(1)} = W^{(1)}[C_{1234}] + W^{(1)}[C_{1243}] + W^{(1)}[C_{1324}]$

Results

- Tree-level prefactor missing (as in YM)
    - expected
- Relative normalisation between IR singular and finite parts incorrect by a factor of - 2
    - 2 from overcounting cusp contributions in W; minus sign more difficult to explain
- Result gauge dependent (but very close to correct one...)

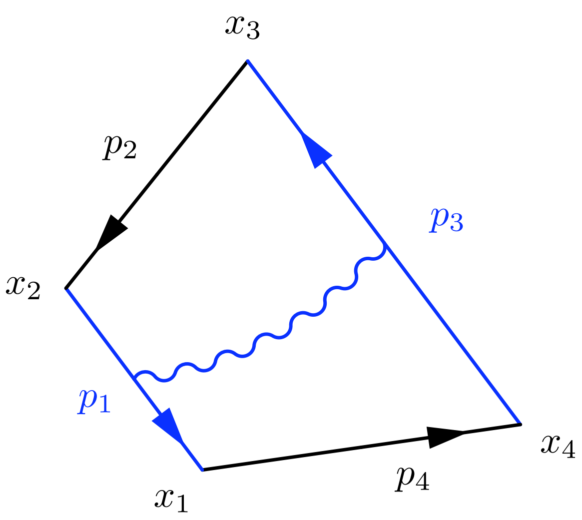
$< W >$ at one loop

- Diagrammatics identical to YM case.
  2 classes of diagrams:



Graviton stretched between two edges meeting at a cusp

A. Infrared divergent



Graviton stretched between two non-adjacent edges

B. Infrared finite

- From diagrams in class A (after summing over permutations):

$$\kappa^2 \frac{c(\epsilon)}{\epsilon^2} \left[(-s)^{1-\epsilon} + (-t)^{1-\epsilon} + (-u)^{1-\epsilon}\right]$$

  - leading divergence cancels due to $s + t + u = 0$
  - subleading term proportional to expected $1/\epsilon$ term:

$$\mathcal{M}^{(1)}\Big|_{\mathrm{IR}} = c_\Gamma \left(\frac{\kappa}{2}\right)^2 \frac{2}{\epsilon} \left[s \log(-s) + t \log(-t) + u \log(-u)\right]$$

- From diagrams in class B:

$$\kappa^2\, c(\epsilon)\, \frac{u}{2}\, \frac{1}{4} \Big[\log^2 \left(\frac{s}{t}\right) + \pi^2 \Big]$$

  - finite part of zero-mass box function
  - sum over all permutations reproduces finite part of amplitude, to all orders in ϵ

$$\mathcal{M}_4^{(1)} \;=\; -i\, s\, t\, u \left(\frac{\kappa}{2}\right)^2 \Big[\mathcal{I}_4^{(1)}(s,t) + \mathcal{I}_4^{(1)}(s,u) + \mathcal{I}_4^{(1)}(u,t) \Big]$$

Q: can we entangle IR divergent and finite contributions ?

Conformal gauge for Gravity

(Brandhuber, Heslop, Nasti, Spence, GT)

- As in YM, it is the gauge where cusp diagrams vanish
- In this gauge, we obtain the correct N=8 supergravity amplitude, to all orders in ϵ
- Special case of a de Donder gauge fixing:

$$\mathcal{L} = -\tfrac{1}{2}(\partial_\mu h_{\nu\rho})^2 + (\partial_\nu h^\nu{}_\mu)^2 + \tfrac{1}{2}(\partial_\mu h^\lambda{}_\lambda)^2 + h^\lambda{}_\lambda \partial_\mu \partial_\nu h^{\mu\nu}$$

Free Lagrangian of linearised gravity

$$\mathcal{L}^{(\mathrm{gf})} = \frac{\alpha}{2}\left(\partial_\nu h^\nu_\mu - \frac{1}{2}\partial_\mu h^\alpha_\alpha\right)^2$$

- α-gauge fixing
- $\alpha = -2$ usual de Donder gauge
- $\alpha = -\frac{2\epsilon}{1+\epsilon}$ conformal gauge

- Graviton propagator in configuration space:

$$\Delta^{\rm conf}_{\mu\nu,\mu'\nu'}(x) \sim \frac{\epsilon+1}{\epsilon}\left[\frac{1}{(-x^2)^{1+\epsilon}}\left(\eta_{\mu'(\mu}\eta_{\nu)\nu'} + \frac{\epsilon}{2(\epsilon+1)^2}\eta_{\mu\nu}\eta_{\mu'\nu'}\right) + 2\frac{1}{(-x^2)^{2+\epsilon}}x_{(\mu}\eta_{\nu)(\nu'}x_{\mu')}\right]$$

- Cfr. gluon propagator in configuration space:

$$\Delta^{\rm conf}_{\mu\nu}(x) \;\sim\; \frac{\epsilon+1}{\epsilon}\frac{1}{(-x^2+i\varepsilon)^{1+\epsilon}}\left[\eta_{\mu\nu} \;-\; 2\frac{x_\mu x_\nu}{x^2}\right]$$

$$J_{\mu\nu}(x) := \eta_{\mu\nu} \;-\; 2\frac{x_\mu x_\nu}{x^2}$$ Inversion tensor

Summary

- Not quite same iterative structure in N=8 supergravity as in N=4 super Yang-Mills
    - uniform transcendentality of the result
    - finite remainder is relatively simple
    - IR divergences cancel
- Wilson loop almost reproduces amplitude
    - Gauge-dependent expression
    - Result closely related to correct answer
    - Conformal gauge

Can we do better ?

- Wilson loop calculation in N=4 super Yang-Mills
    - Agreement with MHV amplitudes in N=4
    - Can we understand why MHV amplitudes and Wilson loops are related ?
    - Can we extend this to non-MHV amplitudes ?