Factorization, process-dependence and Universality

A Perspective on Gauge Theory Scattering

Workshop on Gauge Theory
and String Theory ETH, Zurich, July 4, 2008

George Sterman

- I. How we get away with perturbative QCD
- II. Factorization and Resummation
- III. The Classic Case: Q_T resummation
- IV. Poles in Color Exchange Amplitudes
- V. Finding the Basic Exchange (webs)

I. How we get away with perturbative QCD

The sorrows of QCD perturbation theory:

1. Confinement

$$\int \mathrm{e}^{-iq\cdot x} \langle 0 |\, T[\phi_a(x)\ldots]\, |0\rangle$$

   has no $q^2 = m^2$ pole for any field (particle) ϕ_a that transforms nontrivially under color (confinement)

2. The pole at $p^2 = m_\pi^2$

$$\int \mathrm{e}^{-iq\cdot x} \langle 0 |\, T[\pi(x)\ldots]\, |0\rangle$$

   is not accessible to perturbation theory (χSB etc., etc.)

- And yet we use infrared safety & **asymptotic freedom**:

$$
\begin{aligned}
Q^2\ \hat{\sigma}_{\rm SD}(Q^2,\mu^2,\alpha_s(\mu)) &= \sum_n c_n(Q^2/\mu^2)\ {\alpha_s}^n(\mu) + \mathcal{O}\left(1/Q^p\right) \\
&= \sum_n c_n(1)\ {\alpha_s}^n(Q) + \mathcal{O}\left(1/Q^p\right)
\end{aligned}
$$

- What are we really calculating? PT for color singlet operators
  - $\int \mathrm{e}^{-iq\cdot x}\langle 0|\, T[J(x)J(0)\dots]\,|0\rangle$ for color singlet currents

    $\mathrm{e}^+\mathrm{e}^-$ total, sum rules etc. "no scale"

– Another class of color singlet matrix elements:

EEC (1978) . . . Sveshnikov and F. V. Tkachov (1996), Korchemsky, Oderda, GS (1997)

. . . Bauer, Fleming, Lee, GS (08) Hofmann & Maldacena (08)

$$\lim_{R\to\infty} R^2 \int dx_0 \int d\hat{n}\, f(\hat{n})\, \mathrm{e}^{-iq\cdot y} \langle 0|\, J(0) T[\hat{n}_i T_{0i}(x_0, R\hat{n}) J(y)]\, |0\rangle$$

With T_{0i} the energy momentum tensor

– These are what we really calculate: jet cross sections, etc.

If the "weight" $f(\hat{n})$ introduces no new dimensional scale,
and all $d^k f / d\hat{n}^k$ bounded, then individual final states
have IR divergences, but these cancel in sum over collinear
splitting/merging & soft parton emission
because they respect energy flow.

We regularize these divergences dimensionally (typically) and “pretend” to calculate the long-distance enhancements only to cancel them in infrared safe quantities

It is this intermediate step that makes the calcualtions tough, and is part [not all] of why higher-order calculations are hard!

Resummation organizes large, or potentially large, terms from high orders in α_s at the short-distance scale.

II. Factorization and Resummation

$$Q^2\sigma_{\rm phys}(Q,m) = \omega_{\rm SD}(Q/\mu, \alpha_s(\mu)) \otimes f_{\rm LD}(\mu, m) + \mathcal{O}\left(1/Q^p\right)$$

– $\mu =$ **factorization scale**; $m=$ **IR scale** (m *may* be perturbative)

– **New physics** in $\omega_{\rm SD}$; $f_{\rm LD}$ "universal"

– ep DIS inclusive, $\rm pp \rightarrow$ jets, $Q\bar{Q}$, $\pi(p_T)$ . . .

- **Whenever there is factorization, there is evolution**

$$0 = \mu \frac{d}{d\mu} \ln \sigma_{\rm phys}(Q, m)$$

$$\mu \frac{d \ln f}{d\mu} = -P(\alpha_s(\mu)) = -\mu \frac{d \ln \omega}{d\mu}$$

PDF f <u>or</u> Fragmentation D

- **Wherever there is evolution there is resummation**

$$\ln \sigma_{\rm phys}(Q, m) = \exp \left\{ \int_q^Q \frac{d\mu'}{\mu'} P\left(\alpha_s(\mu')\right) \right\}$$

- Infrared safety & factorization proofs:
  - (1) $\omega_{\rm SD}$ **incoherent** with long-distance dynamics
  - (2) Mutual incoherence when $v_{\rm rel} = c$:
    Jet-jet factorization Ward identities.
  - (3) Wide-angle soft radiation sees only total color flow:
    jet-soft factorization Ward identities.
  - (4) Dimensionless coupling and renormalizability
    $\Leftrightarrow$ no worse that logarithmic divergence in the IR:
    fractional power suppression $\Rightarrow$ finiteness

III. The Classic Case: Q_T resummation

Every final state from a hard scattering carries the imprint of QCD dynamics from at all distance scales

- Look at transverse momentum distribution at order α_s

$$q(p_1) + \bar{q}(p_2) \to \gamma^*(Q) + g(k) \,,$$

- Treat this $2 \to 2$ process at lowest order (α_s) "LO" in factorized cross section, so that $\mathbf{k} = -\mathbf{Q}_T$

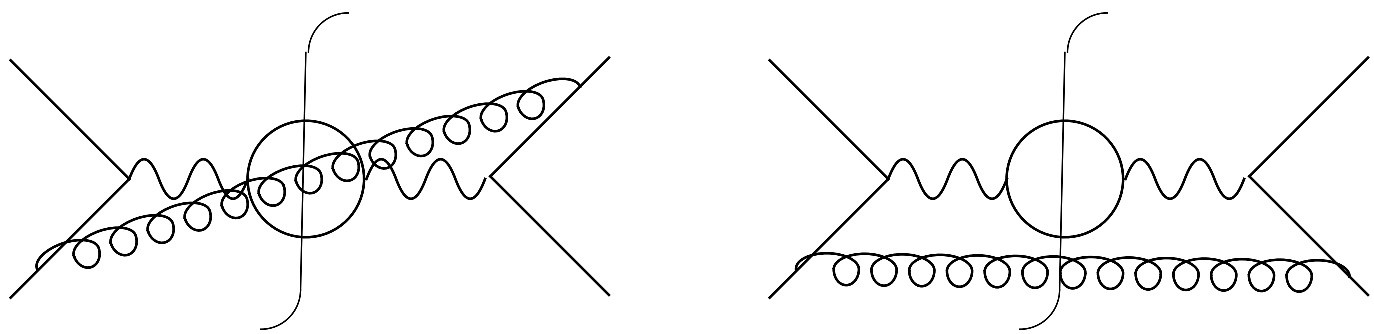
- **Factorized cross section at fixed $\mathbf{Q}_T$:**

$$\frac{d\sigma_{NN\to\mu^+\mu^-+X}(Q,p_1,p_2)}{dQ^2d^2\mathbf{Q}_T} = \int_{\xi_1,\xi_2}\sum_{a=q\bar{q}}\frac{d\hat{\sigma}_{a\bar{a}\to\mu^+\mu^-(Q)+X}(Q,\mu,\xi_1p_1,\xi_2p_2,\mathbf{Q}_T)}{dQ^2d^2\mathbf{Q}_T}$$
$$\times\, f_{a/N}(\xi_1,\mu)\, f_{\bar{a}/N}(\xi_2,\mu)$$

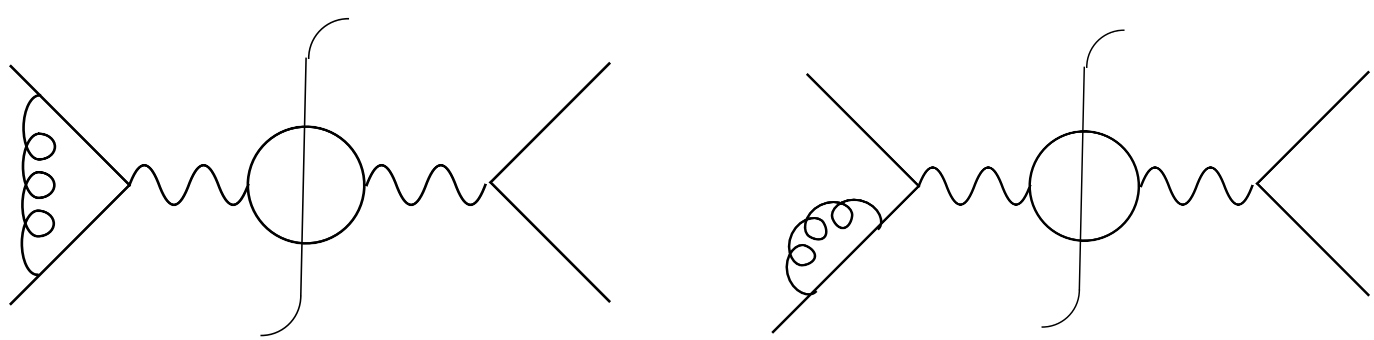
- μ is the factorization scale that separates IR (f) from UV ($d\hat{\sigma}$) in quantum corrections.

- The diagrams at order α_s. Finite for $\mathbf{Q}_T \neq 0$ . . .

Gluon emission contributes at $Q_T \neq 0$



Virtual corrections contribute only at $Q_T = 0$



$$\frac{d\hat{\sigma}^{(1)}_{q\bar{q}\to\gamma^* g}}{dQ^2\, d^2\mathbf{Q}_T} = \sigma_0 \frac{\alpha_s C_F}{\pi^2} \left(1 - \frac{4\mathbf{Q}_T^2}{(1-z)^2 \xi_1 \xi_2 S}\right)^{-1/2}$$

$$\times \left[\frac{1}{\mathbf{Q}_T^2}\frac{1+z^2}{1-z} - \frac{2z}{(1-z)Q^2}\right]$$

as long as $\mathbf{Q}_T \neq 0$, $z = Q^2/\xi_1\xi_2 S \neq 1$.

Q_T integral $\to \frac{\ln(1-z)}{1-z}$; z integral $\to \frac{\ln \mathbf{Q}_T^2}{\mathbf{Q}_T^2}$.

The leading singularity in $\mathbf{Q}_T$

- z integral: If Q^2/S not too big, PDFs nearly constant:

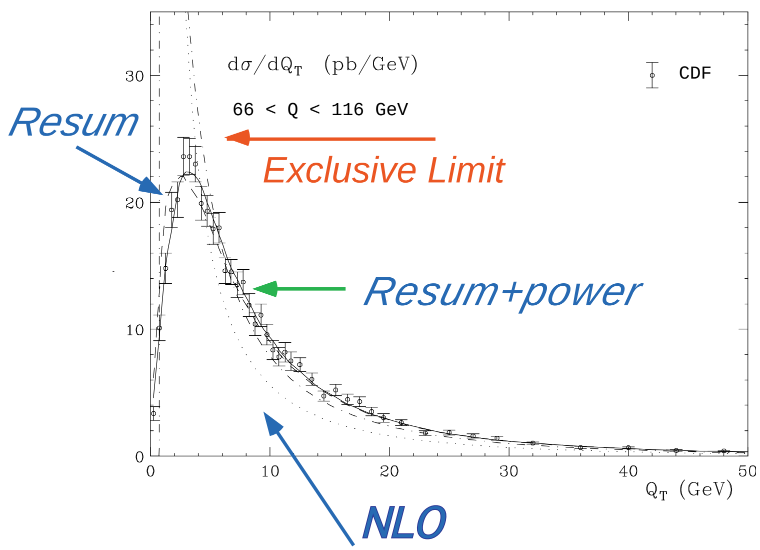
$$\frac{1}{\mathbf{Q}_T^2} \int_{1-Q^2/S}^{1-\mathbf{Q}_T^2/Q^2} \frac{dz}{1-z} = \frac{1}{\mathbf{Q}_T^2} \ln\left[\frac{Q^2}{\mathbf{Q}_T^2}\right]$$

$\Rightarrow$ Prediction for Q_T dependence:

$$\frac{d\sigma_{NN\to\mu^+\mu^-+X}(Q,\mathbf{Q}_T)}{dQ^2\,d^2\mathbf{Q}_T} = \frac{\alpha_s C_F}{\pi}\,\frac{1}{\mathbf{Q}_T^2}\,\ln\left[\frac{Q^2}{\mathbf{Q}_T^2}\right]$$

$$\times \sum_{a=q\bar{q}} \int_{\xi_1\xi_2} \frac{d\hat{\sigma}_{a\bar{a}\to\mu^+\mu^-(Q)+X}(Q,\mu)}{dQ^2}\, f_{a/N}(\xi_1,\mu)\, f_{\bar{a}/N}(\xi_2,\mu)$$

- Compare to: Z p_T (from Kulesza, G.S., Vogelsang (2002))



- $\ln Q_T/Q_T$ works pretty well for large Q_T

- But at smaller Q_T reach a maximum, then a decrease near "exclusive" limit (parton model kinematics)

- Most events are at "low" $Q_T \ll Q = m_Z$.

Getting to $Q_T \ll Q$: Transverse momentum resummation

(Logs of Q_T)/Q_T to all orders

How? Variant factorization and separation of variables

q and $\bar{q}$ "arrive" at point of annihilation with transverse momentum of radiated gluons in initial state.

q and $\bar{q}$ radiate independently (fields don't overlap!).

Final-state QCD radiation too late to affect cross section

$$\frac{d\sigma_{NN\to\mu^+\mu^-+X}(Q,\mathbf{Q}_T)}{dQ^2 d^2\mathbf{Q}_T}$$

Summarized by: Q_T-factorization:

$$\frac{d\sigma_{NN\to QX}}{dQd^2Q_T} = \int d\xi_1 d\xi_2\ d^2\mathbf{k}_{1T} d^2\mathbf{k}_{2T} d^2\mathbf{k}_{sT}\, \delta\left(Q_T - k_{1T} - k_{2T} - k_{sT}\right)$$
$$\times\ H(\xi_1 p_1, \xi_2 p_2, Q, n)_{a\bar{a}\to Q+X}$$
$$\times \mathcal{P}_{a/N}(\xi_1, p_1\cdot n, k_{1T})\, \mathcal{P}_{\bar{a}/N}(\xi_2, p_2\cdot n, k_{2T})\ U_{a\bar{a}}(k_{sT}, n)$$

The $\mathcal{P}'s$: new **Transverse momentum-dependent** PDFs

Also need U: "soft function" for wide-angle radiation

Symbolically:

$$\frac{d\sigma_{NN\to QX}}{dQ\,d^2Q_T} = H \times \mathcal{P}_{a/N}(\xi_1, p_1\cdot n, k_{1T})\,\mathcal{P}_{\bar{a}/N}(\xi_2, p_2\cdot n, k_{2T})$$
$$\otimes_{\xi_i, k_{iT}}\, U_{a\bar{a}}(k_{sT}, n)$$

We will **solve** for the k_T dependence of the $\mathcal{P}$'s.

New factorization variables: n^μ apportions gluons k:

$$p_i\cdot k < n\cdot k \;\Rightarrow\; k \;\in \mathcal{P}_i$$
$$p_a\cdot k, p_{\bar{a}}\cdot k \;>\; n\cdot k \;\Rightarrow\; k \in U$$

Convolution in $k_{i,T}$s $\Rightarrow$ Fourier $\mathrm{e}^{i\vec{Q}_T\cdot\vec{b}}$

The factorized cross section in "impact parameter space":

$$\frac{d\sigma_{NN\to QX}(Q,b)}{dQ} = \int d\xi_1 d\xi_2$$
$$\times\ H(\xi_1 p_1, \xi_2 p_2, Q, n)_{a\bar{a}\to Q+X}$$
$$\times \mathcal{P}_{a/N}(\xi_1, p_1\cdot n, b)\, \mathcal{P}_{\bar{a}/N}(\xi_2, p_2\cdot n, b)\ U_{a\bar{a}}(b,n)$$

Now we can resum by separating variables!

the LHS independent of $\mu_{\rm ren}$, n $\Rightarrow$ two equations

$$\mu_{\rm ren}\frac{d\sigma}{d\mu_{\rm ren}} = 0 \quad n^\alpha\frac{d\sigma}{dn^\alpha} = 0$$

Method of Collins and Soper, and Sen (1981)

Change in jet must cancel change in (UV) H and (IR) U:

$$p \cdot n \frac{\partial}{\partial p \cdot n} \ln \mathcal{P}(p \cdot n/\mu, b\mu) = G(p \cdot n/\mu) + K(b\mu)$$

G matches H, K matches U. Renormalization indep. of n^μ:

$$\mu \frac{\partial}{\partial \mu} \left[\, G(p \cdot n/\mu) + K(b\mu) \,\right] = 0$$

$$\mu \frac{\partial}{\partial \mu} G(p \cdot n/\mu) \; = \; A(\alpha_s(\mu)) \; = \; - \, \mu \frac{\partial}{\partial \mu} K(b\mu)$$

Solve this one first. μ in α_s varies (& α_s need not be small).

$$G(p\cdot n/\mu) + K(b\mu) = G(p\cdot n/\mu) + K(\mu/p\cdot n) - \int_{1/b}^{p\cdot n} \frac{d\mu'}{\mu'} A_a(\alpha_s(\mu'))$$

The consistency equation for the jet becomes

$$p\cdot n \frac{\partial}{\partial p\cdot n} \ln \mathcal{P}(p\cdot n/\mu, b\mu) = G(p\cdot n/\mu) + K(\mu/p\cdot n) - \int_{1/b}^{p\cdot n} \frac{d\mu'}{\mu'} A(\alpha_s(\mu'))$$

Integrate $p\cdot n$ and get double logs in $b \to \alpha_s^n \frac{\ln^{2n-1}(Q/Q_T)}{Q_T}$.

Transformed solution back to Q_T: all the (Logs of Q_T)/Q_T,

Which fits the data; (viz. Yuan, Nadolsky et al.; Ellis, Veselli; Kulesza, Stirling)

$$\frac{d\sigma_{NN\mathrm{res}}}{dQ^2\, d^2\vec{Q}_T} = \sum_a H_{a\bar{a}}(\alpha_s(Q^2)) \int \frac{d^2b}{(2\pi)^2}\, e^{i\vec{Q}_T\cdot\vec{b}} \exp\left[\, E_{a\bar{a}}^{\mathrm{PT}}(b,Q,\mu)\,\right]$$

$$\times \sum_{a=q\bar{q}} \int_{\xi_1\xi_2} \frac{d\hat{\sigma}_{a\bar{a}\to\mu^+\mu^-(Q)+X}(Q,\mu)}{dQ^2}\, f_{a/N}(\xi_1, 1/b)\, f_{\bar{a}/N}(\xi_2, 1/b)$$

"Sudakov" exponent links large and low virtuality:

$$E_{a\bar{a}}^{\mathrm{PT}} = -\int_{1/b^2}^{Q^2} \frac{dk_T^2}{k_T^2} \left[2A_q(\alpha_s(k_T))\, \ln\left(\frac{Q^2}{k_T^2}\right) + 2B_q(\alpha_s(k_T))\right]$$

With $B = 2(K+G)_{\mu=p\cdot n}$, and lower limit: $1/b$ (NLL)

IV. Poles in Color Exchange Amplitudes

- What distinguishes hadron colliders.

- Multiloop scattering amplitudes in dimensional regularization

  (Catani (1998) Tejeda-Yeomans & GS (2002) Kosower (2003) Aybat, Dixon & GS (2006))

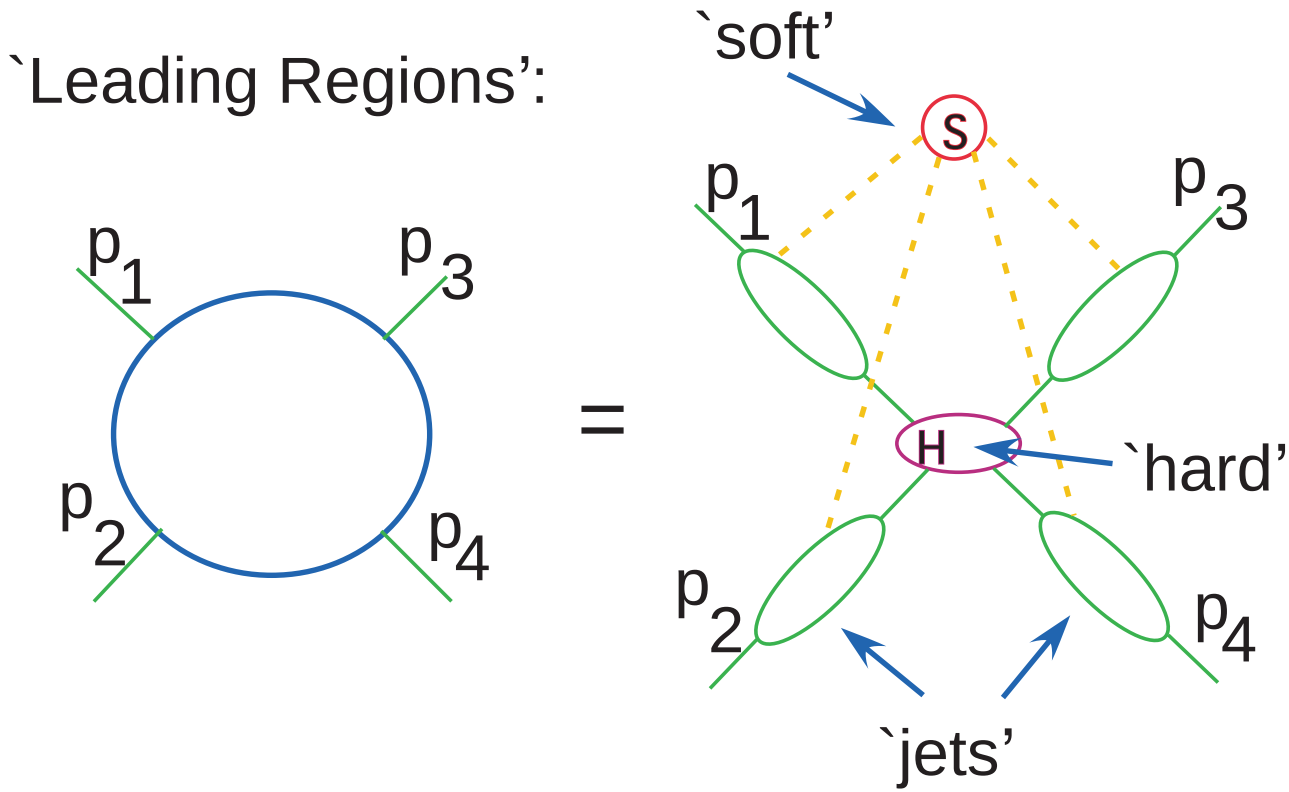
  – Amplitude for partonic process

$$\mathrm{f}: \quad f_A(p_A, r_A) + f_B(p_B, r_B) \to f_1(p_1, r_1) + f_2(p_2, r_2)$$

$$\mathcal{M}^{[\mathrm{f}]}_{\{r_i\}}\left(p_j, \frac{Q^2}{\mu^2}, \alpha_s(\mu^2), \epsilon\right) = \mathcal{M}^{[\mathrm{f}]}_{L}\left(p_j, \frac{Q^2}{\mu^2}, \alpha_s(\mu^2), \epsilon\right)\, (c_L)_{\{r_i\}}$$

- Need to control poles in ϵ for factorized calculations at fixed order and for resummation.

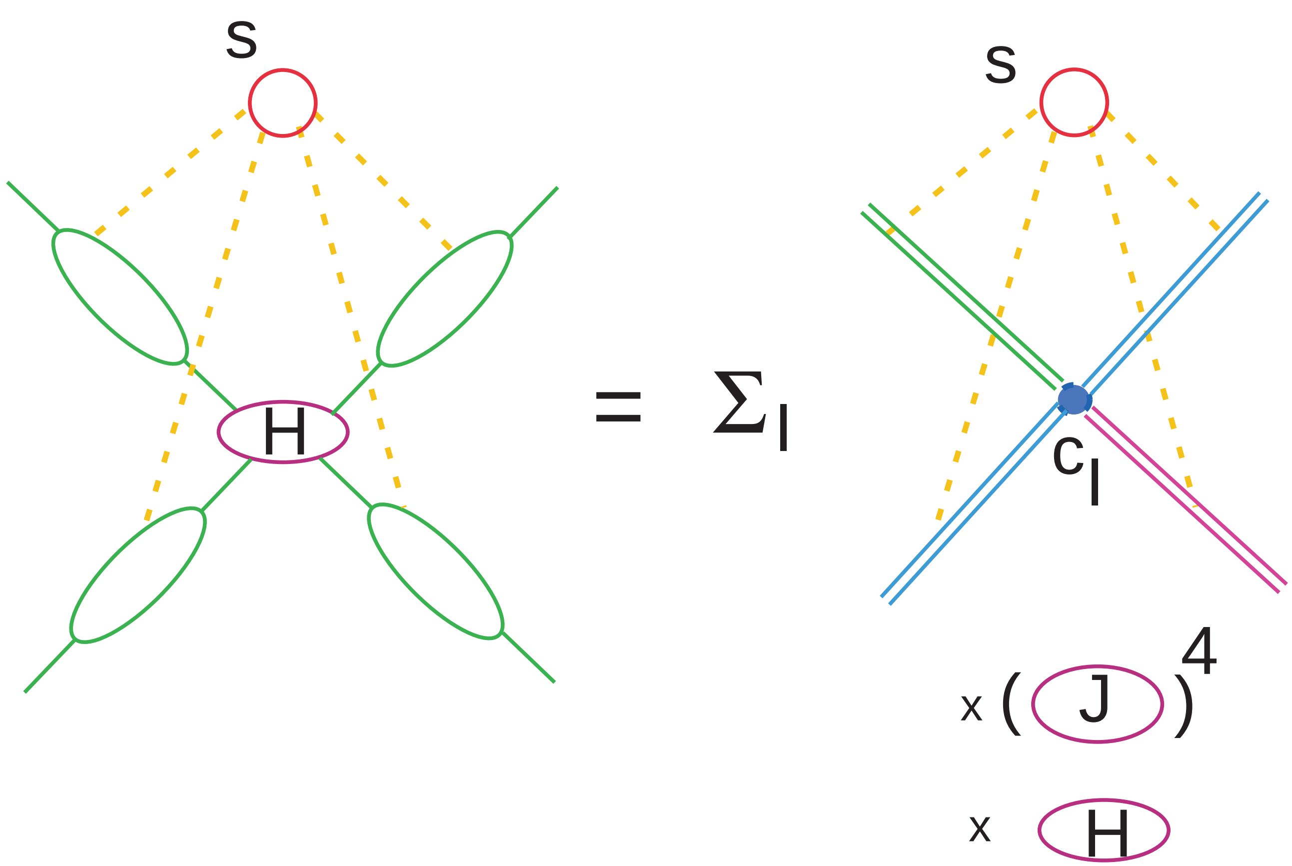
- Source of double logs and poles in dimensional reg.:



- The same cast of characters as for Q_T.

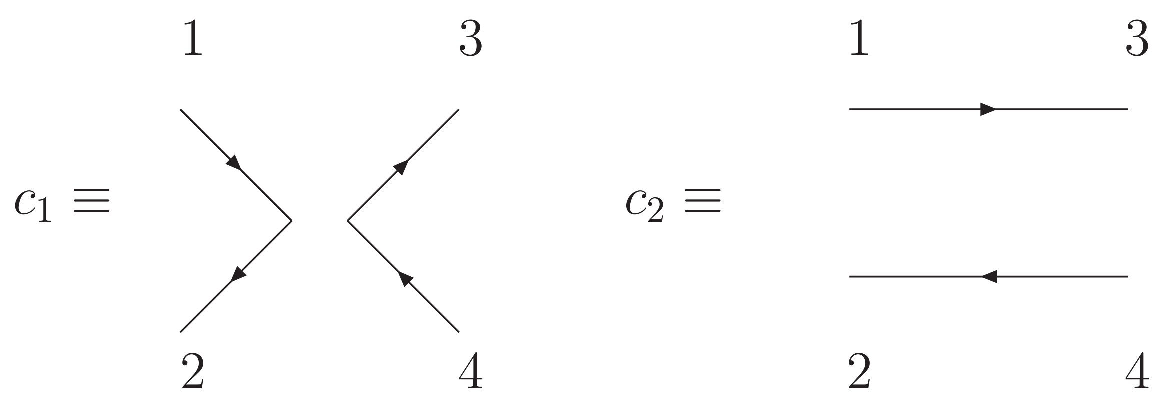
- Same separation (Sen (1983)):

Factorization of soft gluons:



- $\varepsilon = 2 - d/2$ plays the role of b !

- Example of c_I: $q\bar{q}$ tensors $(c_L)_{\{r_i\}}$:



  – Jet/soft factorization for amplitude. :

$$\mathcal{M}_L^{[\mathrm{f}]}\left(p_i, \frac{Q^2}{\mu^2}, \alpha_s(\mu^2), \epsilon\right) = \prod_{i=A,B,1,2} J_i^{[\mathrm{virt}]}\left(\frac{Q^2}{\mu^2}, \alpha_s(\mu^2), \epsilon\right)$$

$$\times \mathbf{S}_{LI}^{[\mathrm{f}]}\left(p_i, \frac{Q^2}{\mu^2}, \alpha_s(\mu^2), \epsilon\right) h_I^{[\mathrm{f}]}\left(\wp_i, \frac{Q^2}{\mu^2}, \alpha_s(\mu^2)\right)$$

- Special case: “0$\longrightarrow$ 2, $\Gamma_{\rm singlet}$. Once again, factorize:

$$\Gamma_{\rm singlet} = H(p_i \cdot n/\mu)\; S(\alpha_s(\mu),\epsilon)\; \prod_{i=1,2} J(p_i \cdot n/\mu, \epsilon)$$

- Same reasoning, boost invariance plus scale variation . . .

- Gives an evolution equation for $\Gamma_{\rm singlet}(Q)$

$$\begin{aligned}
Q^2 \frac{\partial}{\partial Q^2} \log\left[\Gamma\left(\frac{Q^2}{\mu^2},\alpha_s(\mu^2),\epsilon\right)\right] &= \frac{1}{2}\left[K\left({\rm e},\alpha_s(\mu^2)\right) + G\left(\frac{Q^2}{\mu^2},\alpha_s(\mu^2),\epsilon\right)\right] \\
&= K\left(\epsilon,\alpha_s(\mu^2)\right) + G\left(-1,\overline{\alpha}\left(\frac{\xi^2}{\mu^2},\alpha_s(\mu^2),\epsilon\right),\epsilon\right) \\
&\quad + \frac{1}{2}\int_{\xi^2}^{\mu^2} \frac{d\lambda^2}{\lambda^2}\gamma_K\left(\overline{\alpha}\left(\frac{\lambda^2}{\mu^2},\alpha_s(\mu^2),\epsilon\right)\right)
\end{aligned}$$

With K, G, γ_K separation constants as above.

– Jet function: $J = \sqrt{\Gamma_{\mathrm{singlet}}(Q^2)}$ (Tejeda-Yeomans & GS (2002))

– Soft function labelled by color exchange (singlet, octet . . .)

– Factors require dimensional regularization

– Same factorization $\longrightarrow$ resummation

– Poles at 2- and higher loops . . .

– Relation to supersymmetric Yang-Mills theories

(Anastasiou, Bern, Czakon, Dixon, Kosower & Smirnov (2006) N=4)

Scale variation $\Rightarrow$ scale invariance; otherwise reasoning unchanged.

– The dimensionally-regularized jets from $\Gamma_{\rm singlet}$:

(Magnea & GS (1990), after Mueller, Collins & Soper, Sen (1980 - 81))

$$J_i\left(\frac{Q^2}{\mu^2},\alpha_s(\mu^2),\epsilon\right) = \exp\left\{ \frac{1}{4}\int_0^{-Q^2} \frac{d\xi^2}{\xi^2}\left[K^{[i]}(\alpha_s(\mu^2),\epsilon)\right.\right.$$
$$+G^{[i]}\left(-1,\bar{\alpha}_s\left(\frac{\mu^2}{\xi^2},\alpha_s(\mu^2),\epsilon,\right)\epsilon\right)$$
$$\left.\left.+\frac{1}{2}\int_{\xi^2}^{\mu^2}\frac{d\tilde{\mu}^2}{\tilde{\mu}^2}\gamma_K^{[i]}\left(\bar{\alpha}_s\left(\frac{\mu^2}{\tilde{\mu}^2},\alpha_s(\mu^2),\epsilon\right)\right)\right]\right\}.$$

$$J_i\left(\frac{Q^2}{\mu^2},\alpha_s(\mu^2),\epsilon\right) = \exp\left[\sum_{n=1}^{\infty}\left(\frac{\alpha_s(\mu^2)}{\pi}\right)^n \sum_{n=1}^{n+1}\frac{E_m^{[i]\,(n)}(\varepsilon)}{\varepsilon^n} + \text{finite}\right]$$

– Double poles: γ_K, K exactly as $A \leftrightarrow \Gamma_{\rm cusp}$

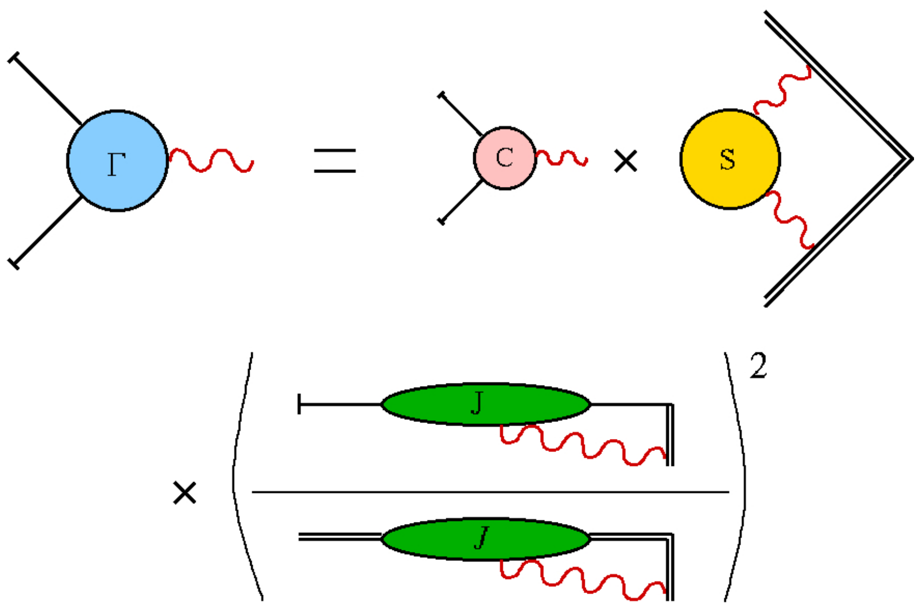
. . . just as in N-4; Alday & Maldacena (2007), but w/ $\mathcal{N}=4$ A

– Single poles from G (Dixon, Magnea, GS (2008))

– G also generates finite coefficient of poles in $\Gamma_{\rm singlet}$

(Moch, Vermaseren, Vogt, 2005)

– Rederive by once again factoring the form factor



– where the "singlet product of Wilson lines"

$$\mathcal{S}\left(\beta_1 \cdot \beta_2, \alpha_s(\mu^2), \mathrm{e}\right) = \langle 0| \Phi_{\beta_2}(\infty, 0)\, \Phi_{\beta_1}(0, -\infty)\, |0\rangle$$

obeys (Korchemsky & Marchesini (1993); Belitsky (1998))

$$\mu \frac{d}{d\mu} \log \mathcal{S}\left(\alpha_s(\mu^2), \varepsilon\right) = -G_{\mathrm{eik}}\left(\alpha_s(\mu^2)\right) + \frac{1}{2}\int_0^{\mu^2} \frac{d\xi^2}{\xi^2} \gamma_K\left(\overline{\alpha}(\xi^2, \varepsilon)\right)$$

– G_{eik}: non-collinear poles that cancel in evolution kernel.

- The full G for the form factor is:

$$G = 2B + G_{\text{eik}} + \beta(g)\frac{\partial}{\partial g}C(\alpha_s(Q))$$

- Same combination noted in DIS & Drell Yan by
  Idilbi, Ji, Yuan (2007) Becher, Neubert, Pecjak (2007)
  Becher, Neubert, Xu (2007)

– Dimensionally-regularized S

(Tejeda-Yeomans & GS (2002))

$$\mathbf{S}^{[\mathrm{f}]}\left(\frac{Q^2}{\mu^2}, \alpha_s(\mu^2), \epsilon\right) = \mathrm{P}\ \exp\left[-\frac{1}{2}\int_0^{-Q^2}\frac{d\tilde{\mu}^2}{\tilde{\mu}^2}\mathbf{\Gamma}^{[\mathrm{f}]}\left(\bar{\alpha}_s\left(\frac{\mu^2}{\tilde{\mu}^2}, \alpha_s(\mu^2), \epsilon\right)\right)\right]$$

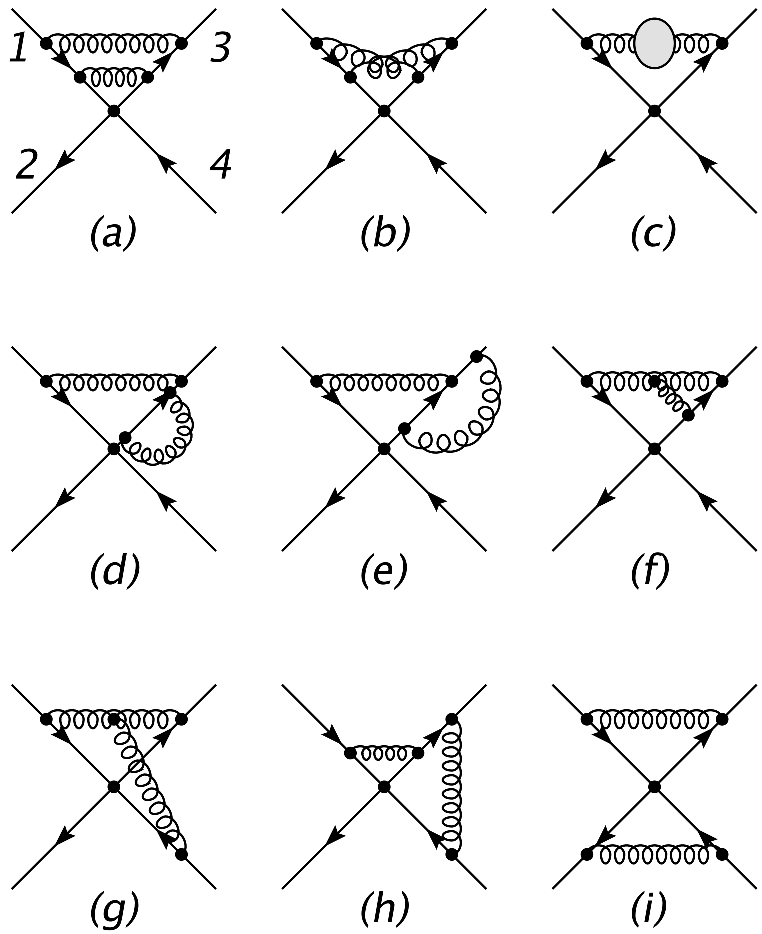
$\Gamma^{[\mathrm{f}]}$: anomalous dimension; color mixing

- New result for all massless $2 \to n$ processes (Aybat, Dixon, GS (2006))

$$\Gamma_S = \frac{\alpha_s}{\pi} \left(1 + \frac{\alpha_s}{\pi} K \right) \Gamma_{S'}^{(1)} + \cdots$$

$\Gamma^{(2)} = (K/2)\Gamma^{(1)}$ with same K as in the DGLAP splitting.

Related to the "CMW" or MC/bremsstrahlung scheme.
(Catani, Marchesini & Webber (1990))



The diagrams with 3g vertices vanish!

To NNLO, “single-web” exchange generalizes single gluon.
(C.F. Berger, 2002)

- The full two-loop single-pole terms $\times$ LO are simply

$$\left[\sum_{i \in \mathrm{f}} \frac{E_1^{[i]\,(2)}}{\varepsilon} + \frac{1}{4\varepsilon} \Gamma_S^{[\mathrm{f}]\,(2)} \right] \times \ \mathrm{LO}$$

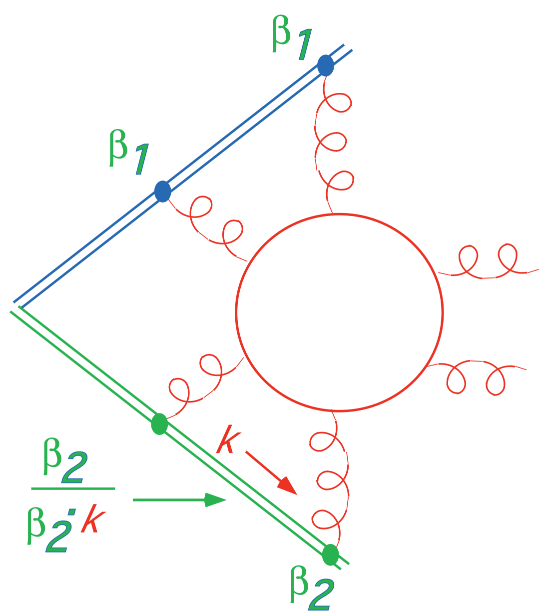
- $E_1^{[i]\ (2)}$ is 2 loop single pole in Sudakov form factor
(Ravindran, Smith, van Neerven (2005))

Agrees with Jantzen, Kuhn, Penin, Smirnov (2005, 2006) in EW logs.

- Hints of unexpected simplicity for IR gluons.

V. FINDING THE BASIC EXCHANGE

- Look for more insight into the
- "independent emission" – exponentiation – by analogy to soft photons.
- A typical diagram (for final-state sources)



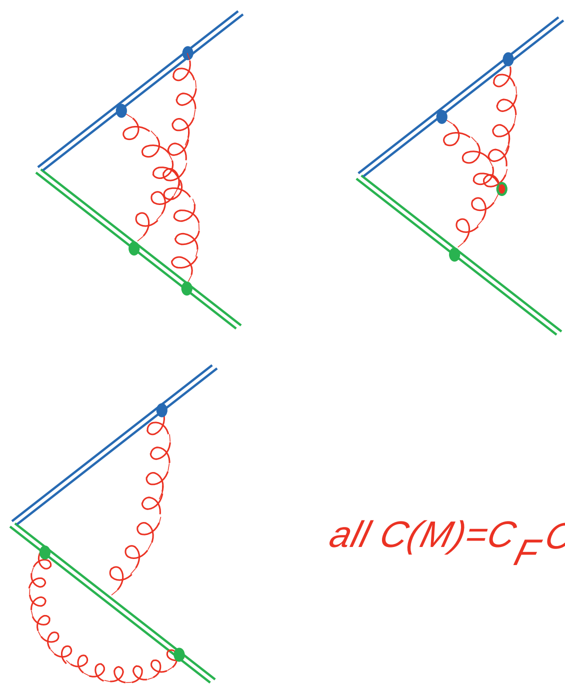
- Webs and exponentiation for soft contributions to weighted ($e = Q_T$, 1-thrust, etc.) cross sections (GS; Gatheral, Frenkel and Taylor, 1981)

$$\frac{d\sigma}{de_a} = \sum_{n=0}^{\infty} \frac{1}{n!} \int de\ \delta(e - e_s)\ \otimes_{\sum e_i = e_a} \prod_{i=1}^{n} E(e_i)$$

$$E(e) = \sum_{\text{states n}} \mathcal{W}_n(e) = \sum_{\mathcal{M}} C(\mathcal{M}_n)\, \mathcal{M}_n^2(e)$$

- The $\mathcal{M}_n^2$ are momentum integrals.

- The $C(\mathcal{M}_n)$ are modified color factors for $\mathcal{M}_n$s.
  Examples at α_s^2:



all $C(M)=C_F C_A$

- Notice that non-planar diagrams contribute in $N_c \longrightarrow \infty$ limit!

- The webs determine exponentiation under transforms:

$$\tilde{S}_e(N) \equiv \int de \, \mathrm{e}^{-Ne} \frac{dS}{de} = \exp\left[\int de' \, \mathrm{e}^{-Ne'} \, E(e')\right]$$

- *Double logarithmic behavior is encoded in the construction of the webs $\mathcal{W}$. Subdivergences cancel.*

- Each web gives a single collinear and infrared logarithm just like a single gluon.

- In a theory with a fixed coupling (SYM . . .) a web would act exactly like a single gluon.

- The 2-loop structure of Γ_S is an intriguing suggestion that "web=NP gluon" could generalize to arbitrary hard processes.

- For some ($e.g.$ DY) cross sections, this gives a very specific template for the all-orders form, up to corrections from recoil:

- Boost invariance in the eikonal annihilation cross section $\Rightarrow$

$$\ln \hat\sigma^{(\mathrm{eik})}(N,Q) = \sum_N \int d\mathrm{PS}_N \; \theta(Q^2-k^2)|M^{(\mathrm{eik})}|^2 \; e^{-Nk_0/Q}$$

$$= \int_0^{Q^2} \frac{\rho(\alpha_s(u,\varepsilon))}{u^2} \; \left[K_0\left(\frac{2Nu}{Q}\right) + \ln\frac{u}{Q}\right]$$

$$+ \ln \bar{N} \int_0^{Q^2} \frac{du^2}{u^2} A(\alpha_s(u,\varepsilon))$$

- The "new" α_{eff}: $\rho(\alpha_s(u,\varepsilon)) = A(\alpha_s(u,\varepsilon)) + \frac{\partial D}{\partial \ln \mu^2}$

- A: "cusp" anomalous dimension;.

Summary

- Have found a key to higher orders in factorization properties of gauge theories.
- Two equations $\leftrightarrow$ boost invariance & scale invariance
- Extends from QCD to supersymmetric variants, and EW (Lipatov, Fadin; Kühn, Penin . . .)
- The basic structure not limited to weak coupling, only calculation of the anomalous dimensions.
- Applications both to cross sections and perturbative S-matrix
- Structure of $\Gamma_S^{(2)}$ suggests a soft-gluon-web relation