Unitarity, Tadpole and Spurious Pole

Bo Feng

ZHEJIANG UNIVERSITY

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I. Motivation

- Last couple year, there are big progresses in the calculation of general one-loop amplitudes
- The speed of calculations depends on following two things:
  - Input expressions: the simpler the better.
  - Algorithm of reductions: the faster the better

Input expressions

- Feynman diagram input: very general but complicated long expressions.
- Off-shell recursion relations: Much better, but not gauge invariant [Berends, Giele, 1988]
- On-shell recursion relation: very compact, but with several points needed to be noticed. [Britto, Cachazo, Feng, Witten '04]
  - Not applicable for all cases, for example, $\lambda\phi^4$ theory.
  - The existence of spurious poles.
  - The dealing of fermions and gauge boson in general *D*-dimension. [Ellis, Giele, Kunszt, Melnikov, 08] [Giele, Kunszt, Melnikov, 08]

Reduction methods:

Last couple years, there are great achievements in the reduction method. We can divide then into following two categories approximately:

- The OPP like methods: [Ossola, Papadopoulos, Pittau] [Ellis, Giele, Kunszt] [Giele, Kunszt, Melnikov]
- The Unitarity cut method: [Bern, Dixon, Dunbar, Kosower] [Britto, Buchbinder, Cachazo, Feng] [Mastrolia] [Anastasiou, Britto, Feng, Kunszt, Mastrolia] [Forde] [Kilgore] [Badger]

OPP like method

The important points of OPP like methods:

- The structure of integrand level expansions, especially the structure of spurious terms.
- The use of general unitarity cut method and the recursive solving algorithm.
- The need to determine both coefficients of basis and coefficients of spurious terms.
- The universal treatment of both massless and massive theories, for example, tadpole coefficients.

Unitarity cut method

- The big point of unitarity cut method is the on-shell tree level input.
- However, the reduction method can be different. For example, using OPP method.
- Here, we will use the unitarity cut method in narrow sense: reading out coefficients by phase space integration. [Bern, Dixon, Dunbar, Kosower]
- Using holomorphic anomaly the general algorithm of phase space integration has been given. [Cachazo, Svrcek, Witten] [Britto, Buchbinder, Cachazo, Feng]

Unitarity cut method

The important points of Unitarity cut method:

- Original is useful for massless only. But it has been generalized to massive case. [Anastasiou, Britto, Feng, Kunszt, Mastrolia]
- We need only to get coefficients of basis, i.e., there is no need for spurious terms.
- We can get bubble coefficient without solving box and triangle first.

Unitarity cut method

However, there are some serious issues for the "narrow" unitarity cut method:

- Unitarity cut method can not be used to determine tadpole coefficients, not complete for massive theory.
- When using on-shell recursion, the existence of spurious poles need to be treated carefully.

Aim of talk

The aim of today's talk:

- Complete the Unitarity cut method for massive theory by given the algorithm to get tadpole coefficients.
- Give the explicit algebraic expressions for tree-level input with spurious poles.

II. Tadpole

Idea

- From OPP method, we know that at integrand level we have following expansion $I = \sum_i (c_i + \widetilde{c}_i(\widetilde{\ell}))I_i$ where $D_0 = \widetilde{\ell}^2 - M_1^2 - \mu^2$, $D_i = (\widetilde{\ell} - K_i)^2 - m_i^2 - \mu^2$ [Ossola, Papadopoulos, Pittau]
- The coefficient of tadpole $\frac{1}{D_0}$ is given by c_{1,D_0}.
- Now we divide an arbitrary $D_K = (\widetilde{\ell} - K)^2 - M_K^2 - \mu^2$ at both sides and get

$$I_{new} = \frac{I}{D_K} = \sum_i (c_i + \widetilde{c}_i(\widetilde{\ell}))\frac{I_i}{D_K}$$

- The key observation is that the wanted tadpole coefficients c_{1,D_0} becomes part of bubble coefficient of I_{new} with basis $\frac{1}{D_0 D_K}$.

Idea

- Now our strategy is clear:
  - Calculate the bubble coefficient after adding D_K. In another word, we are doing the unitarity cut using D_0 and D_K.
  - Decouple other contributions and keep only the original tadpole contribution.
- Then the key step is how to decouple other contributions? To answer this question, we need to understand following two questions:
  - Where do these contributions come from?
  - What are the forms of these extra contributions?

Source of other contributions

- It is obvious that these extra contributions to bubble come from spurious terms in original integrand expansion.
- Let us discuss more explicitly:
  - The box spurious term: It is given by $\frac{Tr(\widetilde{\ell}\ell_1\ell_2 K_3\gamma_5)}{D_0 D_i D_j D_m}$. Adding D_K, we have

$$\frac{Tr(\widetilde{\ell}\ell_1\ell_2 K_3\gamma_5)}{D_0 D_i D_j D_m D_K}$$

which contributes to new box coefficients only, i.e., there is no contribution to bubble D_0, D_K from original box spurious terms.

Source of other contributions

- The triangle spurious term: After adding D_K, we will have following two terms:

$$\frac{(\widetilde{\ell}\cdot\ell_3)^j}{D_0 D_i D_j D_K}, \qquad \frac{(\widetilde{\ell}\cdot\ell_4)^j}{D_0 D_i D_j D_K}, \qquad j = 1, 2, \ldots$$

  which will contribute to bubble $D_0 D_K$ only when $j \geq 2$.
- The bubble spurious term: All original bubble spurious terms have the potential to contribute to bubble $D_0 D_K$.
- The tadpole spurious term: All original tadpole spurious terms have the potential to contribute to bubble $D_0 D_K$.

Form of other contributions

- The triangle spurious term: For $j = 2$ we have contribution

$$\frac{(p_1 \cdot q_1)[(2q_1 \cdot p_1)(2q_1 \cdot q_2) - 2q_1^2(2p_1 \cdot q_2)]}{q_1^2 \Delta(q_1, q_2)} + \{q_1 \to q_2\}$$

- The key observation is the overall factor $(p_1 \cdot q_1)$ and $(p_1 \cdot q_2)$ of these two terms. In fact we have

$$p_1 \cdot q_1 \quad = \quad P_1 \cdot K_1 - \frac{(P_1 \cdot K)(K_1 \cdot K)}{K^2}$$

In our case, K_1 from propagator D_1 and $P_1 = \ell_3$. By OPP construction we have $P_1 \cdot K_1 = 0$, thus to decouple the contribution, we need to put conditions:

$$K_1 \cdot K = 0, \qquad K_2 \cdot K = 0 \qquad (1)$$

Form of other contributions

- We have done similar calculations for other spurious terms and found that under conditions $K_i \cdot K = 0$, almost all contributions have been decoupled except following two:
  - Tadpole spurious term $\frac{(-2\widetilde{\ell}\cdot K)}{D_0 D_K}$ with contribution $-(K^2 + M_1^2 - M_K^2)$.
  - Bubble spurious term

$$\frac{(-2\ell \cdot n)^{2j} - \frac{1}{2j+1}[(-2\ell \cdot K_1)^2 - 4K_1^2 \ell^2]^j}{D_0 D_1}$$

    For example, for $j = 1$ which is the only one we need for renormalizable theory, it is given by $\frac{K_1^2+M_1^2-M_{K_1}^2}{3}$.
- Thus to decouple spurious tadpole terms, we need to add another condition

$$(K^2 + M_1^2 - M_K^2) = 0 \qquad (2)$$

Form of other contributions

- Under above two conditions, have we decoupled all other contributions? Not really!!!
There is still one type spurious term contribution we can not decouple:

$$\frac{(-2\ell\cdot n)^2-\frac{1}{2j+1}[(-2\ell\cdot K_1)^2-4K_1^2\ell^2]}{D_0D_1}$$
$$\to\frac{K_1^2+M_1^2-M_{K_1}^2}{3}$$

- Summary: Under two conditions we can decouple all other spurious term contributions, except one.
So we need to find the coefficient of this particular spurious term.

Finding coefficient

- The key observation is that under two conditions, this particular spurious term $\widetilde{b}_{00}$ has nonzero contribution to triangle $\frac{1}{D_0 D_i D_K}$.
- Thus we can calculate triangle coefficient to see if we can get wanted coefficient $\widetilde{b}_{00}$.
- Now we face a similar problem: the triangle coefficient with D_K added will be the sum of original bubble coefficient plus other spurious term contributions.
- We make similar calculations and find that under two conditions, all other contributions to triangle, except $\widetilde{b}_{00}$, will decouple .

Final Algorithm

- Step A: Calculate all bubble coefficients $c_{2,i}$ of basis $\frac{1}{D_0 D_i}$ in original theory.
- Step B: Calculate all triangle coefficients $C_{3,i}$ of $\frac{1}{D_0 D_i D_K}$ with the added D_K under two decoupling conditions. We have

$$C_{3,i} = c_{2,i} - \frac{1}{3}[\Delta[K_1, M_1, M_2] - 4K_1^2\mu^2]\widetilde{b}_{00,i} \qquad (3)$$

- Step C: Calculate the bubble coefficient C_K of $\frac{1}{D_0 D_K}$ with the added D_K under two decoupling conditions. From this we can have

$$c_1 = C_K - \sum_i \frac{K_i^2 + M_1^2 - M_{K_i}^2}{3}\widetilde{b}_{00,i} \qquad (4)$$

Example

- The expression is

$$\int d\widetilde{\ell}\frac{(-2P\cdot\widetilde{\ell})^2}{(\widetilde{\ell}^2-\mu^2-M_1^2)((\widetilde{\ell}-K_1)^2-\mu^2-M_2^2)}=\int d\widetilde{\ell}\frac{(-2P\cdot\widetilde{\ell})^2}{D_0D_1}$$

- Step A: We find

$$\begin{aligned} c_2 &= \frac{4[K_1^2P^2-(P\cdot K_1)^2]}{3(K_1^2)}\mu^2 \\ &+\frac{(K_1^2+M_1^2-M_2^2)^2}{(K_1^2)^2}\left(\frac{4(P\cdot K_1)^2}{3}-\frac{K_1^2P^2}{3}\right) \\ &+\frac{4M_1^2(K_1^2P^2-(P\cdot K_1)^2)}{3K_1^2} \end{aligned}$$

Example

- Step B: We find

$$C_3 = \frac{-2[(P\cdot K_1)^2 - K_1^2(P^2 - \frac{(P\cdot K)^2}{K^2})]}{K_1^2}\mu^2 + \frac{(K_1^2+M_1^2-M_2^2)^2}{(K_1^2)^2}\left(\frac{3(P\cdot K_1)^2}{2} - \frac{K_1^2(P^2-\frac{(P\cdot K)^2}{K^2})}{2}\right) - \frac{2M_1^2[(P\cdot K_1)^2 - K_1^2(P^2-\frac{(P\cdot K)^2}{K^2})]}{K_1^2}$$

- From c_2, C_3 we can solve $\widetilde{b}_{00}$ by

$$C_3 = c_2 - \frac{(K_1^2+M_1^2-M_2^2)^2 - 4K_1^2M_1^2 - 4K_1^2\mu^2}{3}\widetilde{b}_{00}$$

Example

- Notice that by comparing the coefficients of various power of μ^2 we get two equations for only one variable $\widetilde{b}_{00}$: very nontrivial consistent check

$$\widetilde{b}_{00} = \frac{-1}{2(K_1^2)^2}\left((P\cdot K_1)^2 - K_1^2 P^2 + 3\frac{K_1^2(P\cdot K)^2}{K^2}\right)$$

- Step C: New bubble coefficient is given by

$$C[K] = \frac{-3(P\cdot K_1)^2(K_1^2+M_1^2-M_2^2)}{2(K_1^2)^2} + \frac{(K_1^2+M_1^2-M_2^2)}{2(K_1^2)}(P^2 - \frac{(P\cdot K)^2}{K^2})$$

Example

- From this we can solve

$$
\begin{aligned}
c_1 &= C[K] - \frac{(K_1^2 + M_1^2 - M_2^2)}{3}\tilde{b}_{00} \\
&= \frac{(K_1^2 + M_1^2 - M_2^2)}{3(K_1^2)^2}(P^2 K_1^2 - 4(P \cdot K_1)^2)
\end{aligned}
$$

which is the tadpole coefficient we want to find. It is worth to notice that term $(P \cdot K)$ dropped out in middle: another consistent check.

Other remarks

Some remarks:

- For massive case, we have massless tadpoles as well as other degenerated cases, such as $K^2 = m^2$. In practice these basis can be traded with other basis, i.e, there are relations among these basis.
- In our method, we can calculate these coefficients by keeping K^2 arbitrary in middle steps and set $K^2 = 0$ or $K^2 = m^2$ at the final results.

III. Spurious Pole

Problem

Spurious pole:

- In some recent papers we have given explicit algebraic expression for various coefficients.
- The good point of this method is that
  - (1) we do not need to find coefficients of spurious terms;
  - (2) we do not need to recursive solve equations. They have been solved by our explicit expressions.

Problem

- To derive these results, we used the standard form and make unitarity integration explicitly. In another word, we find these expressions under the assumption that there is no spurious pole in the input.
- However, the most compact tree input does have spurious pole. Then we like to ask: could we find explicit algebraic expression in the presence of spurious poles?

The problem

- The first thought to do so is to do phase-space integrations.
- However, there are two difficulties:
  - We do not know how to write expression into total derivative form with general spurious poles.
  - Even we have able to do so, reading out pole contributions is not so easy.
- Thus we need to have another thought to solve this problem.

New understanding of Box coefficients

To get box coefficient, starting from standard input

$$I = \frac{(K^2)^{n+1}}{\langle\ell|K|\ell]^{n+2}} \frac{\prod_{j=1}^{k+n} \langle\ell|R_j|\ell]}{\prod_{i=1}^{k} \langle\ell|Q_i|\ell]} .$$

we find expressions by following steps:

- (a) Multiplying $\langle\ell|K|\ell]$ and $\langle\ell|Q_i|\ell]$ at I;
- (b) Then replacing $|\ell] \to Q_i |\ell\rangle$.
- (c) After these two pure algebraic replacements we get

$$F_i(\lambda) = (K^2)^{n+1} \frac{\prod_{j=1}^{k+n} \langle\ell|R_j Q_i|\ell\rangle}{\langle\ell|KQ_i|\ell\rangle^{n+1} \prod_{t=1,t\neq i}^{k} \langle\ell|Q_t Q_i|\ell\rangle} .$$

New understanding of Box coefficients

- (d) To go further, we multiply $F_i(\lambda)$ by $\frac{K^2\langle\ell|Q_jQ_i|\ell\rangle}{2\langle\ell|KQ_i|\ell\rangle}$;
- (e) Finally, sum up two terms with $|\ell\rangle \to |P_{ji,1}\rangle$ and $|\ell\rangle \to |P_{ji,2}\rangle$. This is the expression for box coefficient.
- (f) The key observation is that from step (a) to step (e), every step is pure algebraic replacement.

Idea for solving the problem

- With above new understanding we can present our strategy for solving the problem of spurious poles.
- Assuming the general spurious pole is given by

$$S_d = s_0 + \sum_i s_i(-2\widetilde{\ell}\cdot V_{1,i}) + \sum_{i_1,i_2} s_{i_1,i_2}(-2\widetilde{\ell}\cdot V_{2,i_1})(-2\widetilde{\ell}\cdot V_{2,i_2}) + \ldots + \sum_{i_1,\ldots,i_d} s_{i_1,\ldots,i_d}(-2\widetilde{\ell}\cdot V_{d,i_1})(-2\widetilde{\ell}\cdot V_{d,i_2})\ldots(-2\widetilde{\ell}\cdot V_{d,i_d}).$$

Idea for solving the problem

- Then the key observation is that following three forms are equivalent to each other algebraically

$$\mathcal{T}(p)^{I} \equiv \frac{\prod_{j=1}^{l}(-2\widetilde{\ell}\cdot P_j)}{\prod_{i=1}^{k} D_i(\widetilde{\ell})}$$

$$\mathcal{T}(p)^{II} = \sum_{r} \frac{s_{i_1,i_2,\ldots,i_r}(-2\widetilde{\ell}\cdot V_{2,i_1})\ldots(-2\widetilde{\ell}\cdot V_{2,i_r})\prod_{j=1}^{l}(-2\widetilde{\ell}\cdot P_j)}{S_d\prod_{i=1}^{k} D_i(\widetilde{\ell})}$$

$$\mathcal{T}(p)^{III} = \sum_{t} c_t \frac{\prod_i(-2\widetilde{\ell}\cdot P_{t,i})}{S_d\prod_j D_{t,j}(\widetilde{\ell})}$$

Idea for solving the problem

- Since form (I) is the standard form, we can take step (a) to step (e) to get box coefficient.
- Now since form (III) is equivalent to form (I) algebraically and step (a) to step (e) are just algebraic replacements, we can apply step (a)-(e) to form (III) also.
- The result is nothing, but the expression for box coefficient in the presence of spurious poles.

Idea for solving the problem

- Similar understanding and pure algebraic replacement can be applied to get algebraic expression for triangle and bubble coefficients although a little bit more involved.

Final results

Now we can list out the final results:

- Starting from input

$$\mathcal{T}^{(N)}(\widetilde{\ell}) = A_L^{\rm tree}(\widetilde{\ell}) \times A_R^{\rm tree}(\widetilde{\ell}).$$

- Pentagon: It is given by

$$\mathrm{Pen}[K_i, K_j, K_r, K] \quad = \quad \mathcal{T}^{(N)}(\widetilde{\ell}_{(i,j,r)}) \cdot D_i(\widetilde{\ell}_{(i,j,r)}) D_j(\widetilde{\ell}_{(i,j,r)}) D_r(\widetilde{\ell}_{(i,j,r)}).$$

where $\widetilde{\ell}_{(i,j,r)}$ is given by

$$\widetilde{\ell} \to \widetilde{\ell}_{ij} \quad = \quad -\frac{1}{2}\Big[(\alpha^{(q_i,q_j)}(u)\beta - \alpha)K + (\alpha^{(q_i,q_j)}(u) - 1)K^2 \left(\alpha_i \xi_i^{(q_i,q_j)} + \alpha_j \xi_j^{(q_i,q_j)}\right)\Big] - \beta \frac{K^2}{\langle \ell | K | \ell]} \alpha^{(q_i,q_j)}(u) P_{\lambda\widetilde{\lambda}}.$$

Finally results

- Box: It is given by

$$\frac{1}{2}\left(\mathcal{T}^{(N)}(\widetilde{\ell}_{ij})\cdot D_i(\widetilde{\ell}_{ij})D_j(\widetilde{\ell}_{ij})-\frac{\mathrm{Pen}[K_i,K_j,K_r,K]}{D_r(\widetilde{\ell}_{ij})}\right)\Bigg|_{\left\{\begin{array}{lcl}|\ell] & \to & |P_{ji,2}\\ |\ell\rangle & \to & |P_{ji,1}\end{array}\right.}$$
$$+\{P_{ji,1}\leftrightarrow P_{ji,2}\}$$

where

$$\widetilde{\ell}_{ij} = -\beta\frac{K^2}{\langle\ell|K|\ell]}\left([\alpha^{(q_i,q_j)}(u)-1]\frac{q_0^{(q_i,q_j,K)}\cdot P_{\lambda\widetilde{\lambda}}}{(q_0^{(q_i,q_j,K)})^2}q_0^{(q_i,q_j,K)}+P_{\lambda\widetilde{\lambda}}\right)$$
$$-\frac{1}{2}(\beta-\alpha)K,$$

Finally results

- Triangle: The triangle coefficient is

$$\frac{1}{2}\frac{(K^2)^{N+1}}{(-\beta\sqrt{1-u})^{N+1}(\sqrt{-4q_s^2K^2})^{N+1}}\frac{1}{(N+1)!\left\langle P_{s,1}\ P_{s,2}\right\rangle^{N+1}}$$
$$\frac{d^{N+1}}{d\tau^{N+1}}\left(\frac{\langle\ell|K|\ell]^{N+1}}{(K^2)^{N+1}}\mathcal{T}^{(N)}(\widetilde{\ell})\cdot D_s(\widetilde{\ell})\Big|_{\begin{cases}|\ell] & \to & |Q_s|\ell\rangle \\ |\ell\rangle & \to & |P_{s,1}-\tau P_{s,2}\rangle\end{cases}}\right.$$
$$\left.+\{P_{s,1}\leftrightarrow P_{s,2}\}\right)\Big|_{\tau\to 0}$$

where

$$\widetilde{\ell} \;=\; \frac{K^2}{\langle\ell|K|\ell]}\left[-\beta\sqrt{1-u}\left(P_{\lambda\widetilde{\lambda}}-\frac{K\cdot P_{\lambda\widetilde{\lambda}}}{K^2}K\right)-\alpha\frac{K\cdot P_{\lambda\widetilde{\lambda}}}{K^2}K\right].$$

- bubble: Similar expression can be written down.

Example: $A(-,+,+,+,+)$

- With our new method, we recalculate $A(-,+,+,+,+)$ using Mathematica Package provided by Maitre-Mastrolia.
- We get full analytic results including rational term.
- We have checked that rational term we have got matches result given by Bern, Dixon and Kosower.

Example: $A(-,+,+,+,+)$

- Pentagon:

$$\frac{s_{23}^3 s_{45}^3 s_{12} s_{15} s_{34} \langle 1\ 2\rangle \langle 3\ 4\rangle \langle 5\ 1\rangle}{[\langle 2\ 3\rangle (\langle 4|k_1 k_2 k_3 k_1|5\rangle + \langle 4|k_1 k_5 k_2 k_3|5\rangle + \langle 4|k_2 k_3 k_4 k_1|5\rangle)]^3}\mu^2$$

- Box

$$\begin{aligned} c_{[23|4|5|1]} &= \mathrm{Box}[Q_2, Q_3, K_{23}] = -\frac{\langle 1|4\rangle\langle 1|5\rangle[3|2][5|4]^2}{\Delta}(\mu^2)^2 \\ &\quad +(\mu^2\text{-term}) \end{aligned}$$

- Triangle

$$\begin{aligned} c_{[4|51|23]} &= \mathrm{Tri}[Q_2, K_{45}] = \frac{\langle 1|2\rangle\langle 1|4\rangle^2\langle 4|K_{23}|4]}{2\langle 1|5\rangle\langle 2|3\rangle\langle 2|4\rangle^2\langle 3|4\rangle\langle 4|5\rangle}\mu^2 \\ c_{[1|23|45]} &= \mathrm{Tri}[Q_3, K_{45}] = 0 \end{aligned}$$

IV. Conclusion

Conclusion

- In this talk we have discussed how to get tadpole coefficients for massive theory. Thus we have made Unitarity cut method a complete algorithm for both massless and massive theory.
- We have also given algebraic expressions for coefficients in the presence of spurious poles, which are unavoidable for compact input using on-shell recursion relation.

Remark

Let us discuss the numerical implement of our method.

- The $\widetilde{\ell}$ has two variables: the μ^2 and the τ.
- The final answer will be polynomial of μ^2, so we can use discrete Fourier transformation to set μ^2 into root of unit. [Mastrolia, Ossola, Papadopoulos, Pittau,08]
- The τ is used to take derivative and set it to zero at the end. Using Mathematica, the derivative is easy to do, but then we need the analytic input.
- The numerical implement of tadpole algorithm is not so trivial.